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MATHEMATICS

On the Notion of Completeness in Proximity Spaces

by

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Presented by K. KURATOWSKI on May 10, 1956

Let X be a proximity space. Then there exists a family of pseudometrics $\{\varrho_\xi\}_{\xi\in\Xi}$ such that the sets $A, B \subset X$ are near to one another if and only if $\varrho_\xi(A,B)=0$ for every $\xi \in \Xi$ *).

Let

$$\delta_\xi(A)=\sup_{x,y\in A}\varrho_\xi(x,y).$$

Let R be a family of subsets of the space X. The family R is said to satisfy Riesz's condition if for every $\varepsilon>0$ and any $\xi_1,\ldots,\xi_k \in \Xi$ there exists an $A \in R$ such that $\delta_{\xi_1}(A)<\varepsilon,\ldots,\delta_{\xi_k}(A)<\varepsilon$.

Definition 1. A proximity space X is called complete if every centred family of closed sets that satisfies Riesz's condition has a non-empty product.

Definition 2. Let $X^\tau(\tau \in T)$ be proximity spaces and let $\{\varrho_\xi^\tau\}_{\xi\in\Xi_\tau}$ be the family of pseudometrics in the space X^τ. By the product $P_\tau X^\tau$ of these spaces we understand the proximity space the set of whose points is the product of the sets of points of the spaces X^τ, and $\{\varrho_{\xi_1,\ldots,\xi_k}^{\tau_1,\ldots,\tau_k}\}$, where $\tau_1,\ldots,\tau_k$ runs over all finite systems of elements of the set T, $\xi_i \in \Xi_{\tau_i}$ and

$$\varrho_{\xi_1,\ldots,\xi_k}^{\tau_1,\ldots,\tau_k}(x,y)=\varrho_{\xi_1}^{\tau_1}(p^{\tau_1}(x),p^{\tau_2}(y))+\ldots+\varrho_{\xi_k}^{\tau_k}(p^{\tau_k}(x),p^{\tau_k}(y)),\ (x,y,\in P_\tau x^\tau),$$

is a family of pseudometrics **).

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*) A pseudometric ϱ_ξ is a real non-negative function of two variables satisfying the conditions:

(I) $\varrho_\xi(x,x)=0$,
(II) $\varrho_\xi(x,y)=\varrho_\xi(y,x)$,
(III) $\varrho_\xi(x,y)+\varrho_\xi(y,z)\geqslant\varrho_\xi(x,z)$.
By $\varrho_\xi(A,B)$ we understand the number $\inf_{\substack{x\in A\\ y\in B}}\varrho_\xi(x,y)$.

**) $p^\tau(x)$ denotes the τ-th co-ordinate of the point $x \in P_\tau X^\tau$.

THEOREM 1. *If X^τ are complete spaces, then $P_\tau X^\tau$ is a complete space.*

This theorem is an immediate corollary of the following generalization of Tikhonov's theorem:

Let X be a topological space and α a property of the families of subsets of that space. Suppose that if $R \in \alpha$ and $R \subset R'$ then $R \in \alpha$. The space X is termed α-bicompact if every centred family of closed sets that has property α has a non-empty product.

THE GENERALIZED THEOREM OF TIKHONOV. *If X^τ are α^τ-bicompact spaces and in the space $P_\tau X^\tau$ we are given a property α such that the projection of every family $R \in \alpha$ upon the space X^τ has the property α^τ, then the space $P_\tau X^\tau$ is α-bicompact.*

In order to deduce Theorem 1 from the generalized theorem of Tikhonov it suffices to take as α and α^τ the sets of all the families satisfying Riesz's condition.

Definition 3. Let X be a proximity space and T an arbitrary set. By X^T we understand the proximity space whose elements are all bounded functions *) defined in the set T with values from X, and the pseudometrics are defined in the following way:

$$\varrho_\xi^*(f,g) = \sup_{t \in T} \varrho_\xi(f(t),g(t)).$$

The following theorems are valid:

THEOREM 2. *If X is a complete space, then X^T is also a complete space.*

THEOREM 3. *If T is a topological space, then the set of functions continuous at the point $t_0 \in T$ is closed in the space X^T.*

In particular the set of all continuous functions is closed in the space X^T; thus if X is a complete space, then the set in question is also a complete space.

Similarly we can introduce the space 2^X of all closed and bounded **) sets of spaces X; namely we assume the following family of pseudometrics in that space:

$$\varrho_\xi^*(A,B) = \max \{\sup_{y \in B} \varrho_\xi(A,y), \sup_{X \in A} \varrho_\xi(x,B)\}.$$

It is not known whether the completeness of X implies the completeness of the space 2^X.

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*) A function f is called bounded if $\delta_\xi(f(\tau)) < \infty$ for every ξ.

**) A set A is said to be bounded if $\delta_\xi(A) < \infty$ for every ξ.

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MATHEMATICS

On Universal Spaces

by

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According to the classical result of Tikhonov, every completely regular space may be topologically imbedded in the Cartesian product of a certain number of unit intervals. An analysis of the proof of Tikhonov's theorem allows us to establish a necessary and sufficient condition for a space Y to be imbedded in a product of a certain number of spaces X. From this condition follow as simple consequences some known "imbedding theorems"; moreover, a new result concerning T_1-spaces can also be obtained from this condition.

I. NOTATIONS. If X is a topological space and T an arbitrary set of the power $\mathfrak{m}$, then $X^{\mathfrak{m}}$ denotes the Cartesian product $\underset{\tau\in T}{P} X_\tau$ (where $X_\tau = X$) with the ordinary Tikhonov topology. If $T_0 \subset T$, then $P_{T_0}(x)$ denotes the projection of the point $x \in X^{\mathfrak{m}}$ on the set $\underset{\tau\in T_0}{P} X_\tau$.

An elementary neighbourhood in $X^{\mathfrak{m}}$ determined by neighbourhoods $U_{\tau_1}, \ldots, U_{\tau_k} \subset X$ (i. e. the set $\underset{x\in X^{\mathfrak{m}}}{E}\,(p_{\tau_i}(x) \in U_{\tau_i})$) will be denoted as $\langle U_{\tau_1}, \ldots, U_{\tau_k}\rangle$.

The unit interval will always be denoted $<0,1>$.

D denotes the T_2-space consisting of two isolated points 0 and 1.

F denotes the T_0-space consisting of the points 0 and 1, in which the only proper closed set is the set $\{1\}$.

$L_{\mathfrak{m}}$ denotes the T-space which contains $\mathfrak{m}$ points and in which only finite sets and the whole space are closed.

II. THEOREM 1. *A space Y may be topologically imbedded in the product of a certain number of spaces X if, and only if, for every closed set $A \subset Y$ and a point $y \bar{\in} A$ there exists a continuous function f which maps Y into X^n (n is finite, and depends on the function f), such that $f(y) \bar{\in} \overline{f(A)}$.*

Proof. Sufficiency. Let $\mathfrak{R} = \{f_\tau\}_{\tau\in T}$ be the family of all functions mentioned in Theorem 1, and let X_τ be the space of the values of the function f (i. e. $X_\tau = X^n$). The product $\underset{\tau\in T}{P} X_\tau$ is evidently homeomorphic

to $X^{\mathfrak{m}}$, and it suffices to show that Y may be imbedded in $\underset{\tau\in T}{P}X_\tau$. Let h be the mapping of Y into $\underset{\tau\in T}{P}X_\tau$ defined by the equality

$$h(y)=x \quad \text{where} \quad P_\tau(x)=f_\tau(y).$$

We shall show that h is a homeomorphism. Since f is continuous for every $\tau\in T$, h is also continuous. If y_1 and y_2 are two distinct points of Y, then the closure of one of these points, say y_1, does not contain the other point, so there exists $\tau\in T$ such that $f_\tau(y_2)\notin \overline{f_\tau(\{y_1\})}$, and thereby $f_\tau(y_1)\neq f_\tau(y_2)$. Hence, $h(y_1)\neq h(y_2)$, i. e. h is a one-to-one mapping. Let A be an arbitrary closed subset of Y and X an arbitrary point of $h(Y)$ belonging to $\overline{h(A)}$. Let $y=h^{-1}(x)$. If $y\in A$, then there exists $\tau\in T$ such that $f_\tau(y)\notin f_\tau(A)$. This means that there exists a neighbourhood $U_\tau\subset X_\tau$ containing $f_\tau(y)$ and disjoint with $f_\tau(A)$. Then the neighbourhood $\langle U_\tau\rangle\subset \underset{\tau\in T}{P}X_\tau$ contains x and is disjoint with $h(A)$, but this contradicts $x\in\overline{h(A)}$. It follows that $y\in A$ and $x=h(y)\in h(A)$, and therefore $h(A)$ is closed in $h(Y)$, but this means that h is a closed mapping and therefore h is a homeomorphism.

Necessity. Let h be a homeomorphism of Y into $\underset{\tau\in T}{P}X_\tau=X^{\mathfrak{m}}$ (where $X_\tau=X$ and $\overline{T}=\mathfrak{m}$). Let A be a closed subset of Y and let $y\in A$. Since h is a closed and one-to-one mapping, then $h(y)\notin\overline{h(A)}$. It follows that there exists a neighbourhood $\langle U_{\tau_1},\dots,U_{\tau_k}\rangle$ of $h(y)$, disjoint with $h(A)$. Let $T_0=\{\tau_1,\dots,\tau_k\}$. The neighbourhood $\langle U_{\tau_1},\dots,U_{\tau_k}\rangle$ considered as a neighbourhood in $\underset{\tau\in T_0}{P}X_\tau$ is also disjoint with $P_{\tau_0}(h(A))$, and evidently contains $P_{T_0}(h(y))$ and thereby $P_{T_0}(h(y))\notin P_{T_0}(h(A))$. It follows that $g=P_{T_0}h$ is a continuous function which maps Y into $X^n=P_{\tau\in T_0}X_\tau$ (where $n=\overline{T}$) such that $g(y)\notin\overline{g(A)}$.

III. Consequences of Theorem 1.

Theorem 2. (Tikhonov). *Every completely regular space can be imbedded in $I^{\mathfrak{m}}$ (for a certain $\mathfrak{m}$).*

Proof. It follows immediately from Theorem 1 by the definition of complete regularity.

Theorem 3. (Alexandroff). *Every T_0-space may be imbedded in $F^{\mathfrak{m}}$ (for a certain $\mathfrak{m}$).*

Proof. Let Y be a T_0-space and let $A=\overline{A}\subset Y$ and $y_0\in A$. We set $f(y)=1$ for $y\in A$ and $f(y)=0$ for $y\in Y-A$. Since the inverse of each closed subset of F under f is closed in Y, the mapping f is continuous and evidently $f(y_0)\notin\overline{f(A)}$. By Theorem 1. we infer that Y may be imbedded in $F^{\mathfrak{m}}$.

Theorem 4. (Alexandroff). *Every zero-dimensional space Y (i. e. a T_1-space having a basis of open-closed neighbourhoods) may be imbedded in $D^{\mathfrak{m}}$ (for a certain $\mathfrak{m}$).*

Proof. If A is a closed subset of Y and $y_0 \bar{\in} A$, then there exists a closed-open set U containing y_0 and disjoint with A. Let us set $f(y)=1$ for $y \in U$ and $f(y)=0$ for $y \in Y-U$. The function f is continuous and $f(y_0) \bar{\in} \overline{f(A)}$ and, by Theorem 1, Y may be imbedded in $D^{\mathfrak{m}}$.

THEOREM 5. *There exists no T_1-space X, such that every T_1-space may be imbedded in $X^{\mathfrak{m}}$. Every T_1-space may be imbedded in $L^{\mathfrak{m}}_{\mathfrak{m}}$ (for a certain $\mathfrak{m}$).*

Proof. Let us suppose that there exists such a space X and let $\mathfrak{n}=\overline{\overline{X}}$, $p > \max\{\mathfrak{n}, \aleph_0\}$. Since L_p can be imbedded in $X^{\mathfrak{m}}$ then by Theorem 1 we infer that there exists a non-constant continuous function f which maps L into X^n (n is finite). Since f is non-constant, we have $f^{-1}(x) \neq L_p$ and therefore $f^{-1}(x)$ is finite for every $x \in X^n$. But $\sum_{x \in X^n} f^{-1}(x) = L_p$, which contradicts $p > \max\{\mathfrak{n}, \aleph_0\}$.

Let Y be an arbitrary T_1-space and let $\mathfrak{p}=\overline{\overline{Y}}$. If $A=\bar{A} \subset Y$ and $y_0 \in A$. In that case there exists a function f which maps Y into L_p, such that

$$\begin{aligned} &f(A) \neq x_0 \in L_p, \\ &f(y) \neq y_0 \qquad \text{for} \qquad y \bar{\in} A, \\ &f(y_1) \neq f(y_2) \qquad \text{for} \qquad y_1, y_2 \in A, y_1 \neq y_2. \end{aligned}$$

Obviously, $f(y_0) \bar{\in} \overline{f(A)}$. Since, for every $x \in L_p$, the set $f^{-1}(x)$ is closed (it is either equal to A or consists of at most one point), and since every proper closed subset of L_p is finite, the set $f^{-1}(x)$ is closed for every closed subset X of L_p, i. e. f is continuous. By Theorem 1, Y can be imbedded in $L^{\mathfrak{n}}_p$. Setting $\mathfrak{m}=\max\{\mathfrak{n}, p\}$, we see that Y can be imbedded in $L^{\mathfrak{m}}_{\mathfrak{m}}$.

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MATHEMATICS

On Quasi-Compact Spaces

by

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A completely regular space X is called quasi-compact if every continuous real function defined on X is bounded. It is obvious that a compact space is quasi-compact, and examples are known of quasi-compact spaces which are not compact.

Alexandroff and Urysohn in their *Mémoire sur les espaces topologiques compacts* have introduced the notion of absolute closity, namely, a space X is absolutely closed if in every space $X \cup \{a\}$, obtained from X by the adjunction of a point a, this point is isolated. They have proved that the notions of absolute closity and bicompactness coincide for completely regular (even regular) spaces.

In an analogous manner one may introduce the notion of countable absolute closity, namely, a completely regular space X is called countably absolutely closed if in every completely regular space $X \cup \{a\}$ in which the point a satisfies the first axiom of countability, this point is isolated.

The purpose of the present paper is to show that the notions of countable absolute closity and quasi-compactness coincide for completely regular spaces.

THEOREM 1. *If X is a completely regular space, then X is quasi-compact if, and only if, X is countably absolutely closed.*

Proof. Let us suppose that X is not quasi-compact. Then there exists a non-bounded continuous real function f_0 defined on X. We may assume that $f_0(x) \geqslant 1$ and $\sup_{x \in X} f_0(x) = +\infty$. We add a "new" point a to the space X taking as a basis of neighbourhoods of a the collection $\{U_n\}$, where

$$U_n = \{a\} \cup \underset{x \in X}{E}(f_0(x) > n).$$

It is obvious that $X \cup \{a\}$ is a Hausdorff space and that the point a is not isolated and satisfies the first axiom of countability. We shall show that $X \cup \{a\}$ is completely regular. Let $A = \bar{A} \subset X \cup \{a\}$ and $x_0 \in A$. If

$x_0 \neq a$, then there exists a continuous function φ, defined on X, such that $\varphi(x)=0$ for $x \in A\{a\}$, $\varphi(x_0)=1$ and $0 \leqslant \varphi(x) \leqslant 1$ for $x \in X$. Let

$$\psi(x)=\frac{\varphi(x)\cdot f_0(x_0)}{f(x)}$$

for $x \in X$ and $\psi(a)=0$. It is obvious that ψ is continuous on $X \cup \{a\}$ and $\psi(x)=0$ for $x \in A$ and $\psi(x_0)=1$. If $x_0=a$, then there exists U_n such that $A \cap U_n=0$. Let us set $\psi(x)=\min(1, n/f_0(x)$ for $x \in X$ and $\psi(a)=0$. It is obvious that ψ is continuous on $X \cup \{a\}$ and $\psi(x)=1$ for $x \in A$.

Let us suppose that X is not countably absolutely closed and let $X \cup \{a\}$ be a completely regular space in which a is not isolated and satisfies the first axiom of countability. Let $\{U_n\}$ be a basis of neighbourhoods of a. For every n there exists a continuous function g_n, defined on $X \cup \{a\}$ such that $g_n(x)=1/2^n$ for $x \in X \setminus U_n$, $g_n(a)=0$, $0 \leqslant g_n(x) 1/2^n$ for $x \in X \cup \{a\}$. Let $g_0(x)=\sum_{n=1}^{\infty} g_n(x)$. The function g is continuous on X and $g_0(x) \neq 0$ for $x \in X$ and $\inf_{x \in X} g_0(x)=0$. It follows that $f_0(x)=1/g_0(x)$ is a non-bounded continuous function defined on X.

Let us observe that the notions of quasi-compactness and compactness coincide for normal spaces. In fact, if X is a non-compact normal space, then there exists an enumerable closed subset X_0 homeomorphic to the set N of positive integers. The function f, which takes the value n at the n-th point of X_0, is a continuous function on X_0. By the normality of X the function f may be extended to a continuous function defined on X, and therefore X is not quasi-compact. Thus from Theorem 1 we obtain at once:

THEOREM 2. *If X is a normal space, then X is compact if, and only if, X is countably absolutely closed.*

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MATHEMATICS

On Infinite Positional Games

by

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1. In this paper we state some theorems on games analogous to but more general than the zero-sum two-person games with perfect information. The proofs will be published in a more detailed paper in Fundamenta Mathematicae.

Section **2** is strictly related to the papers of David Gale and F. M. Stewart [1] and Jan Mycielski and A. Zięba [2] ([2] was printed before [1] was available in Poland; the theorems proved and problems formulated in these papers are very similar). Generalisations of some theorems of [1] and all of [2], for more general games than those of [1] and [2], are given in section **2**.

In section **3** a theorem is given on a special infinite game invented by S. Mazur. This gives some information on the necessity of the axiom of choice in the construction of open (= indeterminate) games of this kind.

In section **4** we introduce games which are generalisations of the pursuit. This seems to be the first game-theoretical consideration of such games, except for some more special investigations by H. Steinhaus and A. Zięba.

2. The game $\Gamma(R,f)$. We shall use the following notations:

(2,1) $E_1, E_2, \ldots$ is a sequence of non-empty sets.

(2,2) $\mathfrak{X}$ is a function defined on finite sequences of the form $e_1, e_2, \ldots, e_n$, where $e_i \in E_i$ and $n = 0, 1, 2, \ldots$, the values of which are the symbols $\mathfrak{A}$ and $\mathfrak{B}$.

Now we shall define a *zero-sum two-person game* $\langle A, B, f \rangle$, where A is the set of strategies of a player $\mathfrak{A}$, B is the set of strategies of a player $\mathfrak{B}$ and $f = f(a,b)$ is a pay-off function ($\mathfrak{A}$ chooses an element $a \in A$; independently $\mathfrak{B}$ chooses an element $b \in B$ and $\mathfrak{B}$ pays to $\mathfrak{A}$ the amount $f(a,b)$):

(2,3) A is a non-empty class of functions a defined on all sequences $e_1, e_2, \ldots, e_n$ for which $\mathfrak{X}(e_1, \ldots, e_n) = \mathfrak{A}$ and such that

$$a(e_1, \ldots, e_n) \in E_{n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

B is defined in the same way for these sequences for which $\mathfrak{X}(e_1,\ldots,e_n)=\mathfrak{B}$.

Clearly, to every pair $a,b\,(a\in A,\ b\in B)$ corresponds a unique sequence (play) $\Pi_{ab}=(e_1,e_2,\ldots)$, such that

$$e_{n+1}=a(e_1,\ldots,e_n) \quad \text{if} \quad \mathfrak{X}(e_1,\ldots,e_n)=\mathfrak{A},$$
$$e_{n+1}=b(e_1,\ldots,e_n) \quad \text{if} \quad \mathfrak{X}(e_1,\ldots,e_n)=\mathfrak{B}.$$

(2,4) $R=\{\Pi\colon \Pi\in E_1\times E_2\times\ldots,$ there exist $a\in A,\ b\in B$ such that $\Pi=\Pi_{ab}\}$.

(2,5) f is a real-valued function defined on R.

(2,6) $f(a,b)=f(\Pi_{ab})$.

This game $\langle A,B,f\rangle$ will be denoted by $\Gamma(R,f)$, and by $\Gamma(R,S)$ if f is the characteristic function of a set $S\subset R$.

Now the E_i are to be treated as discrete topological spaces. Then $R\subset E_1\times E_2\times\ldots$ becomes a topological space in the product topology (metrisable, totally disconnected).

THEOREM 1. *If S is a G_δ or an F_σ, then $\Gamma(R,S)$ is a closed *) game.* **)

An equivalent form of this theorem is the following:

THEOREM 2. *If f is of the first class of Baire, then $\Gamma(R,f)$ is an ε-closed* ***) *game.* ****)

Now we suppose that $E_1,E_2,\ldots$ are any topological spaces and consider the set $R\subset E_1\times E_2\times\ldots$ in product topology.

THEOREM 3. *If R is compact and f continuous, then $\Gamma(R,f)$ is a closed *) game.*

3. The game $\langle\mathcal{A},\mathcal{B}\rangle$. We define a game of type $\Gamma(R,S)$:

(3,1) E_i $(i=1,2,\ldots)$ is the set of all closed intervals with non-empty interiors, contained in $\langle 0,1\rangle$.

(3,2) $$\mathfrak{X}(e_1,\ldots,e_n)=\begin{cases}\mathfrak{A} & \text{for} \quad n=0,2,4,\ldots\\ \mathfrak{B} & \text{for} \quad n=1,3,5,\ldots\end{cases}$$

(3,3) A is the class of all functions satisfying (2,3) and such that

$$a(e_1,\ldots,e_n)\subset e_1\cap e_2\cap\ldots\cap e_n \qquad (n=1,2,\ldots)$$

if the interior of $e_1\cap\ldots\cap e_n$ is non-empty, and otherwise

$$a(e_1,\ldots,e_n)=\langle 0,1\rangle.$$

B is defined in the same way.

(3,4) R is defined as in (2,4).

*) closed = strictly determined with the existence of the best strategies.

**) For countable S and for S belonging to the Boolean Algebra of subsets of R generated by the open sets, this was proved in [1] and [2].

***) ε-closed = strictly determined in the sense of infima and suprema (the best strategies must not exist).

****) For continuous f this was proved in [2].

(3,5) A decomposition of the interval $\langle 0,1\rangle$ into two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ is given and

$$\Pi_{ab} = (e_1, e_2, \ldots) \in S$$

if, and only if,

$$\bigcap_{n=1}^{\infty} e_n \cap \mathcal{A} \quad \text{is non-empty.}$$

This game $\Gamma(R,S)$ will be denoted by $\langle \mathcal{A}, \mathcal{B}\rangle$.

THEOREM 4. *The game $\langle \mathcal{A}, \mathcal{B}\rangle$ is closed to the advantage of player $\mathfrak{A}$ if, and only if, $\mathcal{B}$ is of the first category in some point $p \in \langle 0,1\rangle$. The game $\langle \mathcal{A}, \mathcal{B}\rangle$ is closed to the advantage of player $\mathfrak{B}$ if, and only if, $\mathcal{A}$ is of the first category.* *)

By this theorem if $\mathcal{A}$ or $\mathcal{B}$ has the property of Baire then the game $\langle \mathcal{A}, \mathcal{B}\rangle$ is closed. On the other hand, this theorem enables us to construct decompositions $\mathcal{A}, \mathcal{B}$ of $\langle 0,1\rangle$, such that the game $\langle \mathcal{A}, \mathcal{B}\rangle$ is open (using the axiom of choice).

4. The game $\gamma(R,f)$. We shall use the following notations:

(4,1) P and Q are two metric spaces.

(4,2) $p_0 \in P$ and $q_0 \in Q$ are some fixed points.

(4,3) P_0^T and Q_0^T are non-empty sets of continuous functions defined on $T = \langle 0, \infty)$, taking values in P and Q resp., and such that

1° $p(0) = p_0$ and $q(0) = q_0$ for every $p \in P_0^T$, $q \in Q_0^T$;

2° If $s(t)$ is a function such that for every $t \in T$ there exists a $p \in P_0^T$ $[q \in Q_0^T]$ such that $s(\tau) = p(\tau)$ $[s(\tau) = q(\tau)]$ for every $\tau \in \langle 0,t\rangle$, then $s \in P_0^T$ $[s \in Q_0^T]$.

We shall now define a zero-sum two-person game $\langle A,B,f\rangle$ (cf. section **2**):

(4,4) A is the class of all functions of the form $a_q(t)$, defined for $q \in Q_0^T$, $t \in T$, and such that

1° $a_q \in P_0^T$ for every fixed q.

2° For every $a \in A$ there exists an $\varepsilon_a > 0$, such that for every $t \in T$ the condition

$$q_1(\tau) = q_2(\tau) \quad \text{for} \quad \tau \in \langle 0,t\rangle$$

implies

$$a_{q_1}(\tau) = a_{q_2}(\tau) \quad \text{for} \quad \tau \in \langle 0, t + \varepsilon_a\rangle.$$

B is the class of all functions of the form $b_p(t)$ defined for $p \in P_0^T$, $t \in T$ and satisfying conditions symmetric to 1° and 2°.

It is clear that to every pair a,b corresponds a unique pair $\Pi_{ab} = \langle p,q\rangle$ $(\Pi_{ab}(t) = \langle p(t), q(t)\rangle)$, where $p \in P_0^T$, $q \in Q_0^T$, such that

$$p(t) = a_q(t), \quad q(t) = b_p(t) \quad \text{for} \quad t \in T.$$

*) This theorem was a conjecture of S. Mazur; a proof was announced by S. Banach, but never published (The Scottish Book, Probl. 43, 1935).

(4,5) $R=\{\Pi\colon \Pi\in(P\times Q)^T$, there exist $a\in A$, $b\in B$, such that $\Pi=\Pi_{ab}\}$.

(4,6) f is a real-valued function defined on R.

(4,7) $f(a,b)=f(\Pi_{ab})$.

This game $\langle A,B,f\rangle$ will be denoted by $\gamma(R,f)$.

(4,8) R_1 is the set R with a topology induced by the class of neighbourhoods of the form

$$\{\Pi\colon \Pi\in R,\quad \Pi(\tau)=\Pi_0(\tau)\quad\text{for}\quad \tau\in\langle 0,t\rangle\},\quad\text{where}\quad \Pi_0\in R,\quad t\in T.$$

(4,9) ϱ is a product metric in the space $P\times Q$.

(4,10) R_2 is the set R with the metric

$$r(\Pi',\Pi'')=\min\left[1,\sup_{t\in T}\varrho(\Pi'(t),\ \Pi''(t))\right].$$

Furthermore we suppose that

(4,3*) The functions belonging to R are uniformly and equi-continuous on T in the metric ϱ.

(This is a complement of def. (4,3) — a restriction on the sets P_0^T, Q_0^T which defined A,B and then R).

THEOREM 5. *If f is of the first class of Baire on R_1 and uniformly continuous on R_2, then the game $\gamma(R,f)$ is ε-closed.*

By a natural interpretation of the game of pursuit of a fleet by another fleet on a sea with border and islands, with various pay-off functions, as a game of type $\gamma(R,f)$, we obtain by Theorem 5 that such games have a value.

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On Some Boolean Algebras

by

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Presented by K. KURATOWSKI on May 31, 1937

A non-empty class $\mathfrak{A}$ of subsets of a set X is said to be a σ-field (or an $\aleph_1$-additive Borel field) provided that

(1) if $A_n \in \mathfrak{A}$ for $n = 1, 2, \ldots$, then $\sum_{n=1}^{\infty} A_n \in \mathfrak{A}$;

(2) if $A \in \mathfrak{A}$, then $X - A \in \mathfrak{A}$.

The definitions of an $\aleph_0$-additive field of sets (or any infinite cardinal) is analogous.

A non-empty class $\mathfrak{I}$ of subsets of a set X is said to be an σ-ideal provided that

(1) if $A_n \in \mathfrak{I}$ for $n = 1, 2, \ldots$, then $\sum_{n=1}^{\infty} A_n \in \mathfrak{I}$;

(2) if $A \in \mathfrak{I}$ and $B \subset A$, then $B \in \mathfrak{I}$.

If $\mathfrak{A}$ is a σ-field of sets and $\mathfrak{I}$ is a σ-ideal, then $\mathfrak{A}/\mathfrak{I}$ denotes the σ-complete Boolean algebra obtained from $\mathfrak{A}$ by identification of any sets $A, B \in \mathfrak{A}$ if and only if $(A - B) + (B - A) \in \mathfrak{I}$. The element of $\mathfrak{A}/\mathfrak{I}$ determined by a set $A \in \mathfrak{A}$ will be denoted by (A).

A mapping φ of a σ-complete Boolean algebra $\mathfrak{A}$ into a σ-complete Boolean algebra $\mathfrak{B}$ is said to be a σ-homomorphism provided that

$$\varphi\left(\sum_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \varphi(A_n) \quad \text{and} \quad \varphi(-A) = -\varphi(A) \quad \text{for} \quad A, A_n \in \mathfrak{A}$$

where $-A$ denotes the Boolean complement of A. If there exists a one-to-one σ-homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are said to be isomorphic.

The σ-field of all subsets of a set X will be denoted by ΞX. The σ-field of all Borel subsets of a topological space $\mathfrak{X}$ will be denoted by $\mathfrak{B}(\mathfrak{X})$. The least σ-field containing all sets which are both open and closed in $\mathfrak{X}$ will be denoted by $\mathfrak{M}(\mathfrak{X})$.

A topological space $\mathfrak{X}$ is said to be an absolute $\mathfrak{B}$-space, provided $\mathfrak{X}$ is homogeneous and totally discontinuous, and $\mathfrak{X}$ is a subset of every topo-

pact and totally disconnected space $\mathcal{Y}_0 \supset \mathcal{Y}$ (i. e. there exists a continuous mapping f of $\mathcal{Y}_0$ onto $\mathcal{Y}$ such that $f(y)=y$ for $y \in \mathcal{Y}$).

An example of absolute B-retract is the generalised Cantor discontinuum $\mathcal{C}_{\mathfrak{m}}$ (where $\mathfrak{m}$ is any cardinal), i. e. the Cartesian product of $\mathfrak{m}$ replicas of Hausdorff two-element spaces. Each bicompact, totally disconnected space is homeomorphic to a subset of $\mathcal{C}_{\mathfrak{m}}$ for a suitably great $\mathfrak{m}$. In order that a bicompact, totally disconnected space $\mathcal{Y}$ be an absolute B-retract, it is necessary and sufficient that $\mathcal{Y}$ be a retract of $\mathcal{C}_{\mathfrak{m}}$ for a cardinal $\mathfrak{m}$ (see [2] Theorem 4.1).

This paper treats of Boolean algebras $\boldsymbol{A}$ isomorphic to Boolean algebras of the form $\mathfrak{S}(\mathcal{X})/\boldsymbol{I}$ where $\boldsymbol{I}$ is a σ-ideal of subsets of $\mathcal{X}$. The following theorem characterises these spaces by an extension property of homomorphisms.

(i) *The following conditions are equivalent for an arbitrary σ-complete Boolean algebra $\boldsymbol{A}$:*

(1) *$\boldsymbol{A}$ is isomorphic to a Boolean algebra $\mathfrak{S}(\mathcal{X})/\boldsymbol{I}$, where $\boldsymbol{I}$ is a σ-ideal of subsets of $\mathcal{X}$;*

(2) *for every absolute B-retract $\mathcal{Y}$, every σ-homomorphism h of $\mathfrak{A}(\mathcal{Y})$ into $\boldsymbol{A}$ can be extended to a σ-homomorphism of $\mathfrak{S}(\mathcal{Y})$ into $\boldsymbol{A}$;*

(3) *for every cardinal $\mathfrak{m}$, every σ-homomorphism h of $\mathfrak{A}(\mathcal{C}_{\mathfrak{m}})$ into $\boldsymbol{A}$ can be extended to a σ-homomorphism of $\mathfrak{S}(\mathcal{C}_{\mathfrak{m}})$ into $\boldsymbol{A}$;*

(4) *there is a cardinal $\mathfrak{m} \geqslant \boldsymbol{A}$ such that every σ-homomorphism h of $\mathfrak{A}(\mathcal{C}_{\mathfrak{m}})$ into $\boldsymbol{A}$ can be extended to a σ-homomorphism of $\mathfrak{S}(\mathcal{C}_{\mathfrak{m}})$ into $\boldsymbol{A}$.*

To prove the implication (1) $\Rightarrow$ (2) it suffices to consider only the case where $\boldsymbol{A}=\mathfrak{S}(\mathcal{X})/\boldsymbol{I}$. If h is a σ-homomorphism of $\mathfrak{A}(\mathcal{Y})$ into $\boldsymbol{A}$, then there exists (see [2] Theorem 6.2) a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$ such that

$$h(B)=[\varphi^{-1}(B)] \quad \text{for} \quad B \in \mathfrak{A}(\mathcal{Y}).$$

The σ-homomorphism h_0

$$h_0(A)=[\varphi^{-1}(A)] \quad \text{for} \quad A \in \mathfrak{S}(\mathcal{Y})$$

maps $\mathfrak{S}(\mathcal{Y})$ into $\boldsymbol{A}$ and is an extension of h.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

Suppose that (4) is satisfied. $\mathfrak{A}(\mathcal{C}_{\mathfrak{m}})$ is a free σ-complete Boolean algebra with $\mathfrak{m}$ generators (see [3], Corollary 1). Let g be a mapping of the set of generators of $\mathfrak{A}(\mathcal{C}_{\mathfrak{m}})$ onto $\boldsymbol{A}$. The algebra $\mathfrak{A}(\mathcal{C}_{\mathfrak{m}})$ being free, the mapping g can be extended to a σ-homomorphism h of $\mathfrak{A}(\mathcal{C}_{\mathfrak{m}})$ onto $\boldsymbol{A}$. By (4) the σ-homomorphism h can be extended to a σ-homomorphism h_0 of $\mathfrak{S}(\mathcal{C}_{\mathfrak{m}})$ onto $\boldsymbol{A}$. Let $\boldsymbol{I}$ be the σ-ideal of all sets $A \in \mathfrak{S}(\mathcal{C}_{\mathfrak{m}})$ such that $h_0(A)$ is the zero element of $\boldsymbol{A}$. The Boolean algebra $\boldsymbol{A}$ is isomorphic to $\mathfrak{S}(\mathcal{C}_{\mathfrak{m}})/\boldsymbol{I}$, which proves (1).

(ii) *If a σ-complete Boolean algebra $\boldsymbol{A}$ is isomorphic to a Boolean algebra $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ (where $\boldsymbol{I}$ is a σ-ideal of subsets of $\mathfrak{X}$), then, for every complete separable metric space $\mathcal{Y}$, every σ-homomorphism h of $\mathfrak{B}(\mathcal{Y})$ into $\boldsymbol{A}$ can be extended to a σ-homomorphism of $\mathfrak{S}(\mathcal{Y})$ into $\boldsymbol{A}$.*

It suffices to prove the theorem in the case, where $\boldsymbol{A}=\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$. The space $\mathcal{Y}$ being complete and separable, there exists (see [1] Theorem 4.4) a mapping φ of $\mathfrak{X}$ into $\mathcal{Y}$ such that

$$h(B)=[\varphi^{-1}(B)] \quad \text{for every} \quad B \in \mathfrak{B}(\mathcal{Y}).$$

The homomorphism h_0

$$h_0(A)=[\varphi^{-1}(A)] \quad \text{for} \quad A \in \mathfrak{S}(\mathcal{Y})$$

is the required extension of h.

The problem arises whether the extension property mentioned in (ii) is characteristic for the σ-complete Boolean algebras of the form $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ ($\boldsymbol{I}$ – a σ-ideal). The negative answer is given by the following theorem:

(iii) *There exists a σ-field of sets $\boldsymbol{A}$ such that:*

(1) *$\boldsymbol{A}$ is not isomorphic to a Boolean algebra $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ ($\boldsymbol{I}$ – a σ-ideal);*

(2) *for every separable metric space $\mathcal{Y}$, every σ-homomorphism h of $\mathfrak{B}(\mathcal{Y})$ into $\boldsymbol{A}$ can be extended to a σ-homomorphism of $\mathfrak{S}(\mathcal{Y})$ into $\boldsymbol{A}$.*

Let $\mathfrak{m}=2^{2^{\aleph_0}}$ and let $\boldsymbol{A}$ be the least $2^{\aleph_0}$-additive field (of subsets of $\mathcal{C}_\mathfrak{m}$) containing $\mathfrak{A}(\mathcal{C}_\mathfrak{m})$.

Suppose $\boldsymbol{A}$ is isomorphic to a Boolean algebra $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ ($\boldsymbol{I}$ – a σ-ideal). By (i) the trivial homomorphism h

$$h(B)=B \quad \text{for} \quad B \in \mathfrak{A}(\mathcal{C}_\mathfrak{m}),$$

of $\mathfrak{A}(\mathcal{C}_\mathfrak{m})$ into $\boldsymbol{A}$ can be extended to a σ-homomorphism h_0 of $\mathfrak{S}(\mathcal{C}_\mathfrak{m})$ into $\boldsymbol{A}$. There exists (see [1], Theorem 2.3) a function φ of $\mathcal{C}_\mathfrak{m}$ into itself such that

$$h_0(A)=\varphi^{-1}(A) \quad \text{for} \quad A \in \mathfrak{S}(\mathcal{C}_\mathfrak{m}).$$

In particular,

$$\varphi^{-1}(B)=B \quad \text{for} \quad B \in \mathfrak{A}(\mathcal{C}_\mathfrak{m}),$$

which implies that

$$\varphi(x)=x \quad \text{for each} \quad x \in \mathcal{C}_\mathfrak{m}.$$

Hence

$$h_0(B)=B \quad \text{for each} \quad B \in \mathfrak{S}(\mathcal{C}_\mathfrak{m}).$$

Consequently $\boldsymbol{A}=\mathfrak{S}(\mathcal{C}_\mathfrak{m})$, which is not true since no one-element subset of $\mathcal{C}_\mathfrak{m}$ belongs to $\boldsymbol{A}$. The property (1) is proved.

To prove (2) suppose that $\mathcal{Y}$ is a separable metric space and h is a σ-homomorphism of $\mathfrak{B}(\mathcal{Y})$ into $\boldsymbol{A}$. There exists (see [1] Theorem 2.4) a mapping of $\mathcal{C}_\mathfrak{m}$ into $\mathcal{Y}$ such that

$$\varphi^{-1}(B)=h(B) \in \boldsymbol{A} \quad \text{for} \quad B \in \mathfrak{B}(\mathcal{Y}).$$

If A is any subset of $\mathcal{Y}$, then $\varphi^{-1}(A) \in \boldsymbol{A}$ too. In fact, A belongs to the least $2^{\aleph_0}$-additive field generated by the sets G_n, where $\{G_n\}$ is the open basis of $\mathcal{Y}$. Consequently, $\varphi^{-1}(B)$ belongs to the least $2^{\aleph_0}$-additive field generated by the sets $\varphi^{-1}(G_n) = h(G_n) \in \boldsymbol{A}$. This field is a subfield of $\boldsymbol{A}$, since $\boldsymbol{A}$ is $2^{\aleph_0}$-additive.

Consequently, the formula

$$h_0(A) = \varphi^{-1}(A) \quad \text{for} \quad A \in \mathfrak{S}(\mathcal{Y})$$

defines a σ-homomorphism h_0 of $\mathfrak{S}(\mathcal{Y})$ into $\boldsymbol{A}$ which is an extension of h.

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MATHÉMATIQUE

Remarques sur la méthode de T. Ważewski dans l'étude qualitative des équations différentielles ordinaires

par

A. BIELECKI

Présenté par T. WAŻEWSKI le 17 Avril 1956

Dans la méthode du rétracte, due à T. Ważewski, dans l'étude de l'allure asymptotique des intégrales des équations différentielles ordinaires [3] et dans les variantes postérieures de cette méthode [1], [2] et [4] la notion de *point de sortie stricte* ([3], p. 293) et son rapport à celle de *conséquent* ([3], p. 298) jouent un rôle important. Or, les théorèmes établis dans les mémoires que nous venons de citer peuvent être un peu généralisés si l'on modifie légèrement l'hypothèse qui y intervient et suivant laquelle tous les points de sortie sont des points de sortie stricte. Il suffit, en effet, qu'ils soient des *points de sortie forte*. Cette dernière notion sera définie dans la suite et on verra qu'elle se prête bien aux raisonnements de cette sorte.

Définitions et notations. Nous admettons que les fonctions $f_\nu(t,x_1,\ldots,x_n)$ sont définies et continues dans un ensemble Ω contenu dans l'espace E_{n+1} à $n+1$ dimensions et contenant, à son tour, un ensemble ouvert ω.

Nous admettons encore que par tout point $Q \in \omega$ il passe une seule intégrale du système d'équations différentielles ordinaires

$$\frac{dx}{dt} = f(t,x) \text{ *)}, \tag{1}$$

saturée par rapport à l'ensemble ω.

Soit R un point de sortie par rapport au système (1) et aux ensembles ω et Ω ([3], p. 292). Il sera dit *point de sortie forte* s'il n'existe aucune intégrale

$$x=\varphi(t), \qquad \alpha \leqslant t \leqslant \beta, \qquad (\alpha<\beta)$$

telle que $(\alpha,\varphi(\alpha))=R$ et $(t,\varphi(t)) \in \bar{\omega}$ pour $\alpha \leqslant t \leqslant \beta$. Il est évident que tout point de sortie stricte est un point de sortie forte, mais la proposition inverse serait en défaut, comme le montre l'exemple suivant.

*) Nous utilisons la notation vectorielle.

EXEMPLE. Soit $\Omega = E_{2+1}$, $f_1(t,x,y) = f_2(t,x,y) = 0$ et soient A, B_i et D_i les ensembles définis dans le plan $t=0$ respectivement par les conditions suivantes:

$$(A) \qquad -2 < x < 1, \quad -2 < y < 1,$$

$$(B_i) \qquad -1 \leqslant x \leqslant 1, \quad \frac{1}{2^i} \leqslant y \leqslant \frac{3}{2^{i+1}},$$

$$(D_i) \qquad |x| + \left| y - \frac{3}{2^{i+1}} \right| < \frac{1}{2^{i+2}}.$$

Désignons par $\delta(x,y)$ la distance, dans le plan $t=0$, d'un point (x,y) à l'ensemble complémentaire (dans ce plan) de l'ensemble $C \cdot A - \sum_{i=1}^{\infty} B_i$.

Soient ensuite A_i et I' les ensembles spatiaux définis par les conditions:

$$(A_i) \qquad 0 < t < 2, \quad (x, y+t) \in D_i,$$

$$(I') \qquad (x,y) \in C, \quad 0 < t < \delta(x,y).$$

Soit enfin $\omega = I' + \sum_{i=1}^{\infty} A_i$. On démontre aisément que l'ensemble ω, ainsi défini, est borné, ouvert et homéomorphe à une sphère et que tous les points de sortie par rapport à ω et Ω sont des points de sortie forte, mais ceux du segment: $0 < t < 1$, $x = 0$, $y = -t$, sont des points de sortie et ne sont pas des points de sortie stricte.

LEMME. *Si tout point de sortie est un point de sortie forte, le conséquent $R = C(Q)$ du point variable Q est une fonction continue dans $\omega' = -W + S$, où W désigne l'ombre gauche* ([3], p. 291) *et S l'ensemble des points de sortie.*

Démonstration. Supposons, en effet, qu'une intégrale $x = \varphi(t)$, $\alpha \leqslant t \leqslant \beta$, joigne un point $Q = (\alpha, \varphi(\alpha))$ à son conséquent $C(R) = (\beta, \varphi(\beta)) \in S$. Dans le cas où $Q = C(Q) \in S$ on a $\alpha = \beta$ et l'intégrale se réduit à un seul point Q.

Fixons un entourage $V \subset \Omega$ borné et fermé de cette intégrale et supposons qu'une suite de points $\{Q_i\} \subset \omega'$ converge vers Q. On démontre sans peine qu'il existe une suite partielle $\{Q_{i(\mu)}\}$, $\mu = 1,2,\ldots$, satisfaisant aux conditions suivantes: pour tout $\mu = 1,2,\ldots$ il existe une intégrale $x = \varphi^\mu(t)$, $\alpha^\mu \leqslant t \leqslant \beta^\mu$, telle que $(t, \varphi^\mu(t)) \in V \cdot \omega'$ pour $\alpha^\mu \leqslant t \leqslant \beta^\mu$, $(\alpha^\mu, \varphi^\mu(\alpha^\mu)) = Q_{i(\mu)}$, $T_\mu = (\beta^\mu, \varphi^\mu(\beta^\mu)) \in \mathrm{Front}\,(V \cdot \omega)$ *) et, en outre, $\lim_{\mu \to \infty} \alpha^\mu = \alpha$, $\lim \beta^\mu = \beta$ et la suite d'intégrales $x = \varphi^\mu(t)$ converge presque uniformément vers une intégrale $x = \varphi^*(t)$, $\alpha \leqslant t \leqslant \beta^*$, contenue dans $\overline{\omega} \cdot V$, issue du point Q et aboutissant au point $T = (\beta^*, \varphi^*(\beta^*)) \in \mathrm{Front}\,(V \cdot \omega)$. On aura $\varphi(t) = \varphi^*(t)$ pour $\alpha \leqslant t \leqslant \beta' = \min(\beta, \beta^*)$, puisque les intégrales remplissent la condition d'unicité.

*) C'est-à-dire la frontière de l'ensemble $V \cdot \omega$.

Or, si l'on avait $\beta' = \beta^* < \beta$, le point T appartiendrait à l'intérieur de l'ensemble $V \cdot \omega$; si l'on avait $\beta' = \beta < \beta^*$, l'intégrale $x = \varphi^*(t)$, $\beta \leqslant t \leqslant \beta^*$ serait contenue dans ω. L'un et l'autre étant évidemment exclus, on a $\beta = \beta' = \beta^*$, c'est-à-dire $T = C(Q) = \lim_{\mu \to \infty} T_\mu$. Il existe donc un μ_0 tel que $T_\mu \notin \operatorname{Front}(V)$ et, par suite, $T_\mu = C(Q_{i(\omega)}) \in S$, pour $\mu \geqslant \mu_0$.

Nous avons ainsi constaté que toute suite de conséquents $\{C(Q_i)\}$ contient une suite partielle convergente vers $C(Q)$ lorsque $\lim_{i \to \infty} Q_i = Q$. Il s'ensuit que la fonction $C(Q)$ est continue.

En vertu du lemme que nous venons de démontrer, on peut remplacer l'hypothèse que les points de sortie sont des points de sortie stricte par l'hypothèse, plus faible, qu'ils sont des points de sortie forte dans les théorèmes 1—4 [3], 1 [1], [2], et 4 [4]. On peut aussi modifier d'une manière analogue les autres théorèmes du travail [4]. Le lemme que nous avons démontré nous sera utile dans l'étude des intégrales des équations différentielles dans le cas où l'on ne postule pas que les intégrales satisfont à la condition d'unicité, ce qui fera l'objet d'une communication ultérieure.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

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MATHEMATIQUE

Sur une méthode de régularisation des équations différentielles ordinaires dont les intégrales ne remplissent pas la condition d'unicité

par

A. BIELECKI

Présenté par T. WAŻEWSKI le 17 Avril 1956

Les méthodes d'étude de l'allure asymptotique des intégrales des équations différentielles ordinaires dues à T. Ważewski [5], développées par lui-même [6] et par d'autres auteurs [1], [4], se prêtent non seulement à l'étude des équations différentielles dont les intégrales remplissent la condition d'unicité, mais aussi à celle des équations qui n'ont pas cette propriété.

Dans le premier travail cité on trouve une mention sur ce sujet ([5], p. 313). On y approche le système donné d'équations différentielles par une suite de systèmes plus réguliers jouissant de la propriété d'unicité, on applique à ces derniers la méthode du rétracte et on obtient, sous certaines conditions supplémentaires, par un passage à la limite, les conclusions désirées relatives au système initial. On peut, cependant, éviter ce procédé indirect si l'on sait qu'il existe un système d'équations différentielles identique au système donné en dehors du tube ω [5], satisfaisant à la condition d'unicité dans ω et conservant, sur la frontière de ω, des propriétés qui sont essentielles dans la méthode du rétracte.

Nous allons montrer que ceci est possible sous certaines conditions et nous établirons, dans le cas où l'unicité des intégrales n'est pas postulée, des théorèmes correspondant à ceux de T. Ważewski [5], [6] et A. Pliś [4].

Nous admettrons l'hypothèse suivante.

HYPOTHÈSE (A). *Les fonctions* $f_\nu(t,x_1,\ldots,x_n)$, $\nu=1,2,\ldots,n$, *sont continues dans un ensemble ouvert* Ω *contenu dans l'espace* E_{n+1} *à* $n+1$ *dimensions et contenant un ensemble ouvert* ω *(tube). Nous désignons par* $\Phi=Front\,(\omega,\Omega)$ *la frontière de* ω *par rapport à* Ω; *l'ensemble des points de sortie par rapport*

aux ensembles ω et Ω et au système d'équations différentielles

$$\frac{dx}{dt} = f(t,x) \quad (I)$$

sera désigné par S; l'ensemble engendré par les demi-intégrales asymptotiques à droite (c'est-à-dire saturées à droite par rapport à Ω et contenues dans ω) — par A.

Points de sortie, de sortie forte et de sortie stricte

Nous dirons qu'un point $P_0 \in \Phi$ est un point de sortie s'il existe une demi-intégrale du système (I) négative, issue du point P_0 et contenue dans ω à l'exception du point P_0; nous dirons que P_0 est un point de sortie forte [2] si, de plus, il n'existe aucune demi-intégrale positive, issue du point P_0 et contenue dans $\bar{\omega}$. Le point $P_0 \in \Phi$ s'appellera point de sortie stricte si toute demi-intégrale négative (resp. positive), issue de P_0, contient un segment contenu dans ω (resp. $\Omega - \bar{\omega}$) et aboutissant au point P_0. Les définitions des points d'entrée, d'entrée forte et d'entrée stricte seront analogues. Le point $P_0 \in \Phi$ sera appelé point de glissement extérieur si toute demi-intégrale, issue de P_0, contient un segment aboutissant au point P et contenu dans $\Omega - \bar{\omega}$. L'ensemble de points de sortie sera désigné par S.

Hypothèse (B). *L'ensemble U est contenu dans $\omega+S$, et contient l'ensemble A; l'ensemble U est fermé par rapport à $\omega+S$ et il est saturé à gauche par rapport à $\omega+S$, c'est-à-dire toute demi-intégrale du système (I), issue d'un point de l'ensemble U et saturée par rapport à $\omega+S$, est contenue dans U. L'ensemble $\omega - U$ est non vide. Tous les points de sortie appartenant à $S' = S - U$ sont des points de sortie forte.*

Notations:

$$S'' = S \cdot U,$$
$$T' = \Phi - \bar{U} - S,$$
$$B = \omega - U = \sum_{i=1}^{\infty} B_i,$$

où B_i, $i = 1,2,\ldots$, désignent des ensembles non vides, bornés et fermés, tels que B_i est contenu dans l'intérieur de B_{i+1},

$$T'' = \bar{B} \cdot \bar{U},$$
$$T = T' + T''.$$

*) Nous utilisons la symbolique vectorielle; la formule (I) est une abbréviation de la formule

$$\frac{dx_\nu}{dt} = f_\nu(t, x_1, \ldots, x_n), \qquad \nu = 1,2,\ldots,n.$$

$Z^+(B_i)$ désigne la zone d'émission positive de l'ensemble B_i par rapport à l'ensemble $\bar{B}$, c'est-à-dire l'ensemble engendré par les demi-intégrales positives issues de B_i et saturées par rapport à $\bar{B}$. $S_i = Z^+(B_i) \cdot \Phi \subset S'$ et $S' - \sum_{i=1}^{\infty} S_i$, $Z^+(B_i, \varepsilon)$ désignera l'entourage de l'ensemble $Z^+(B_i)$ de rayon ε.

LEMME 1. *Dans les hypothèses A et B il existe, pour* $\varepsilon_i > 0$, $i = 1,2,\ldots$, *un* $\delta_i > 0$ *satisfaisant à la condition suivante.*

Si $f_\nu^{(i)}(t,x_1,\ldots,x_n)$, $\nu = 1,2,\ldots,n$ *sont des fonctions continues dans* Ω *et si*

$$|f_\nu^{(i)}(t,x_1,\ldots,x_n) - f_\nu(t,x_1,\ldots,x_n)| < \delta_i, \tag{1}$$

la zone d'émission positive $Z_i^+(B_i)$ *de l'ensemble* B_i *par rapport au système d'équations*

$$\frac{dx}{dt} = f^{(i)}(t,x) \tag{I_i}$$

et à l'ensemble $\bar{B}$ *est bornée, fermée et contenue dans l'ensemble* $Z^+(B_i, \varepsilon_i)$; *la zone d'emission positive de l'ensemble* S_i *par rapport à* (I_i) *et* $\bar{B}$ *est disjointe à* B_i.

La démonstration s'appuie sur les théorèmes bien connus en vertu desquels les intégrales dépendent continuellement des fonctions $f_\nu^{(i)}$.

HYPOTHÈSE (C). *L'ensemble* T' *est un ensemble borelien* F_σ. **) Il s'ensuit que* $T' + T'' = T = \sum_{i=1}^{\infty} T_i$, *où les* T_i *sont bornés et fermés et* $T_i \subset T_{i+1}$.

THÉORÈME 1. *Dans les hypothèses A, B et C, si* $\delta(t,x)$ *est une fonction positive et continue dans B, il existe un système de fonctions* $f_1^*(t,x),\ldots,f_n^*(t,x)$ *continues dans* Ω *et satisfaisant aux conditions suivantes:*

1° *Les fonctions* $f_\nu^*(t,x)$ *sont analytiques dans B.*

2° $|f_\nu^*(t,x) - f_\nu(t,x)| < \delta(t,x)$ *dans B.*

3° $f_\nu^*(t,x) \equiv f_\nu(t,x)$ *dans* $\Omega - B$.

4° *Il n'existe pas de demi-intégrales asymptotiques du système d'équations*

$$\frac{dx}{dt} = f^*(t,x) \tag{I*}$$

par rapport aux ensembles B et Ω.

5° *L'ensemble* S^* *des points de sortie par rapport à* (I*), *B et* Ω *est identique à celui des points de sortie forte et il est contenu dans* S'.

6° *L'ensemble des points d'entrée par rapport à* (I*), *B et* Ω *est contenu dans T.*

Démonstration. Nous nous contentons d'exposer l'idée fondamentale menant à la construction d'un système f^*. D'abord, tout en conservant les notations du Lemme 1, nous demandons, pour $i = 1,2,\ldots$, que les

**) Cette hypothèse peut être remplacée par certaines hypothèses plus fortes sur l'allure des intégrales du système (I) au voisinage de l'ensemble $T' + S'$; p. ex. tout point appartenant à $T' + S'$ est un point d'entrée stricte, un point de sortie stricte ou un point de glissement extérieur par rapport à (I), ω et Ω. [6].*

ensembles $Z^{\vdash}(B_i,\varepsilon_i)$ et T_i soient disjoints et nous prenons une fonction $\eta(t,x)$, continue dans $\bar{B}$, nulle sur la frontière de B et remplissant les inégalités $0<\eta(x)<\delta(x)$ dans B, $\eta(t,x)\leqslant\delta_i$, en dehors de l'ensemble B_i. Or, en vertu d'un théorème de H. Whitney (voir [7] p. 76, lemme 6), il existe un système de fonctions $f_\nu^*(t,x)$, $\nu=1,2,\ldots,n$, analytiques dans B, tel que $|f_\nu^*(t,x)-f_\nu(t,x)|<\eta(t,x)$, $\nu=1,2,\ldots,n$ dans B. Nous posons, en outre, $f_\nu^*\equiv f_\nu$ dans $\Omega-B$.

On démontre facilement que les fonctions ainsi définies remplissent bien les conditions 1°—6°.

Lemme 2. *Le conséquent d'un point variable $Q\in B+S'$ par rapport au système d'équations* (I*) *est une fonction continue du point Q.*

La démonstration est analogue à celle du lemme dans [2]. En appliquant la méthode de F. Albrecht et A. Pliś [1], [4], on en déduit le théorème suivant:

Théorème 2. *Dans les hypothèses A, B et C l'ensemble S' des points de sortie par rapport à* (I), *B et Ω est un rétracte par déformation quasi-isotope* [4] *de l'ensemble $B+S'$.*

Connections avec les théorèmes de T. Ważewski et A. Pliś

Supposons, dans l'hypothèse A, que S'' désigne un sous-ensemble fermé par rapport à Φ, de l'ensemble S et V, un ensemble contenu dans $\omega+S'$, où $S'=S-S''$. Soit Z^- la zone d'émission négative de l'ensemble S'' par rapport à $\omega+S$. Supposons encore que l'ensemble $\Phi-S$ soit un F_σ et que tous les points de l'ensemble S' soient des points de sortie forte. Alors nous pouvons admettre que $U=A+Z^-$ et déduire facilement du théorème 2 une proposition analogue à celle de A. Pliś [4]. Les hypothèses relatives à la régularité des équations différentielles sont ici moins restrictives; cependant nous avons introduit certaines restrictions supplémentaires concernant la structure des ensembles $\Phi-S$ et S''. Nous pouvons obtenir pareillement des propositions analogues à celles que T. Ważewski a établi dans son travail [5], p. ex. le théorème suivant:

Théorème 3. *Admettons que l'hypothèse A soit remplie, que l'ensemble $\Phi-S$ soit un F_σ et l'ensemble $S'\subset S$ se compose de points de sortie forte, et que l'ensemble $S''=S-S'$ soit fermé par rapport à S. Dans ces conditions, si*

$$V\subset\omega+S',$$

$V\cdot S'$ est un rétracte de S', et

$V\cdot S'$ n'est pas un rétracte de V,

alors il existe une demi-intégrale positive, issue d'un point de l'ensemble $V\cdot\omega$ et contenue dans $\omega+S''$ qui rencontre l'ensemble S'' ou bien est une intégrale asymptotique à droite.

Les domaines de validité du théorème 1 [5] et de notre théorème 3 se croisent aussi, mais ce dernier est un peu plus fort que la proprosition

que l'on pourrait obtenir en utilisant la méthode proposée par T. Ważewski dans la partie finale de son mémoire ([5], p. 313).

Cas d'un ensemble S' transversal au champ d'éléments linéaires

Nous allons encore énoncer une proposition relative à un cas plus particulier.

THÉORÈME 4. *Admettons que les hypothèses A et B soient remplies et que la distance $\delta(P)$ entre le point P et T soit positive pour $P \in S'$. Admettons de plus que l'ensemble S' soit transversal au champ d'éléments linéaires engendré par le système d'équations* (I). *Cela veut dire qu'il existe, pour tout $P \in S'$, un nombre positif ε et un voisinage σ du point P tels que le contingent de l'ensemble S' au point $Q \in S' \cdot \sigma$ ne contient aucune direction dont l'angle d'inclinaison à l'élément linéaire au point P soit plus petit que ε.*

Dans ces hypothèses les ensembles $B+S'$ et $S' \times I$, où I désigne un intervalle $0 \leqslant \xi < 1$, sont homéomorphes.

REMARQUE 1. Ce théorème est semblable à un théorème établi dans le mémoire [6] de T. Ważewski (théorème 4, p. 147). On peut encore énoncer des propositions analogues aux autres propositions de ce mémoire; je le remets pourtant à une communication que j'ai en vue de faire sur ce sujet.

REMARQUE 2. Je voudrais encore signaler que tous les résultats contenus dans cette note sont valables sans modifications essentielles dans la théorie des équations au paratingent (voir [3] et [8]), ce qui fera aussi l'objet d'une communication ultérieure.

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MATHÉMATIQUE

Certaines propriétés topologiques des intégrales des équations différentielles ordinaires

par

A. BIELECKI

Présenté par T. WAŻEWSKI le 1 Mai 1956

Dans une communication antérieure nous avons énoncé un théorème ([5], Théorème 1) sur la régularisation des équations différentielles, à l'aide duquel nous avons pu adapter, sous certaines conditions supplémentaires, la méthode du rétracte [10], [1], [8] dans l'étude de l'allure des intégrales d'équations différentielles ordinaires au cas où les intégrales ne satisfont pas à la condition d'unicité ([5], Théorème 2 et 3). L'objet de cette note sera un théorème du même genre, ainsi que certaines applications et des rapports de celui-ci avec des recherches plus récentes de T. Ważewski [11].

Hypothèses et notations

Admettons, comme dans [5], que E_{n+1} désigne l'espace à $n+1$ dimensions et que les fonctions $f_\nu(t,x_1,\ldots,x_n)$, où $\nu=1,2,\ldots,n$, sont continues dans un ensemble ouvert $\Omega \subset E_{n+1}$, contenant un ensemble ouvert ω. Nous désignons par $\Phi = \mathrm{Front}(\omega,\Omega)$ la frontière de ω par rapport à Ω et par S l'ensemble des points de sortie ([5] p. 498) par rapport aux ensembles ω et Ω et au système d'équations differentielles

$$\frac{dx}{dt} = f(t,x) \text{ *).} \tag{1}$$

Admettons ensuite qu'un ensemble $U \subset \omega + S$ est fermé par rapport à $\omega+S$ et saturé à gauche par rapport à $\omega+S$ et au système d'équations (1), c'est-à-dire que toute demi-intégrale négative du système (1), issue d'un point de U et contenue dans $\omega+S$, doit être encore contenue dans U. Supposons, en outre, que toute demi-intégrale positive, contenue dans ω et saturée

*) Écrit sous forme vectorielle.

à droite par rapport à Ω, est contenue dans U. Posons $B=\omega-U$, $S'=S-U$, $S''=S\cdot U$, $T'=\Phi-\overline{U}-S$, $T''=\overline{B}\cdot\overline{U}$, $T=T'+T''$ et désignons par $Ct\,(S',Q)$ le contingent de l'ensemble S' au point Q ([6], p. 65), entendu comme un ensemble de directions, par $\sigma(P,\delta)$ la sphère de centre P et de rayon δ, et par $L(P,\varepsilon)$ l'ensemble des directions inclinées sous un angle plus petit que ε sur la tangente au point P à une intégrale du système d'équations différentielles (1).

Admettons enfin

α) que l'ensemble $\overline{T'}\cdot S'\cdot\Omega$ est vide,

β) que pour tout $P\in S'$ il existe un $\varepsilon>0$ et un $\delta>0$ tels que les ensembles $Ct(S',Q)$ et $L(P,\varepsilon)$ sont disjoints si $Q\in S'\cdot\sigma(P,\delta)$.

Il est facile de constater que, dans ces hypothèses, l'ensemble T est fermé par rapport à Ω et l'ensemble S' ne contient que des points de sortie stricte ([5], p. 498) et peut être représenté localement dans un système convenable de coordonnées par l'équation $\tau=F(\xi_1,\xi_2,\ldots,\xi_n)$, où F remplit la condition de Lipschitz.

Théorème I. *Dans les hypothèses que nous venons d'admettre il existe, pour toute fonction $\theta(t,x)$ positive et continue* dans $\Omega^*=\Omega-\overline{U}-T'$, *un système de fonctions $g_\nu(t,x)$, $\nu=1,2,\ldots,n$, continues dans Ω et satisfaisant aux conditions suivantes:*

1. les fonctions $g_\nu(t,x)$ sont analytiques dans Ω^*.

2. $|g_\nu(t,x)-f_\nu(t,x)|\leqslant\theta(t,x)$ pour $\nu=1,2,\ldots,n$ dans Ω^*.

3. $g_\nu(t,x)=f_\nu(t,x)$, pour $\nu=1,2,\ldots,n$ dans $U+T'$.

4. Toute demi-intégrale positive du système d'équations

$$\frac{dx}{dt}=g(t,x), \tag{2}$$

issue d'un point de l'ensemble B et saturée par rapport à Ω passe par un point de S'.

5. S' est l'ensemble des points de sortie par rapport à B,Ω et (2) et satisfait à la condition β) par rapport à (2).

6. L'ensemble des points d'entrée ([5], *p.* 498) *par rapport à B,Ω et* (2) *est contenu dans T'.*

Nous omettons la démonstration qui s'appuie sur un théorème de H. Whitney; tout à fait analogue à celle du Théorème 1 dans [5], elle est même plus facile, puisque les hypothèses α) et β) sont beaucoup plus fortes que celles qui concernent l'ensemble S' dans [5].

Théorème II. *Dans les hypothèses du théorème précédent, les ensembles $B+S'$ et $S\times I$, où I désigne l'intervalle $0\leqslant u<1$, sont homéomorphes; cette homéomorphie fait correspondre à tout point $(t,x)\in S'$ le point $(t,x,0)\in S'\times I$.*

Dans la démonstration, dont je ne donne qu'une idée générale, on profite du fait que les intégrales du système d'équations (2) remplissent la condition d'unicité dans $B+S'$.

On fait d'abord correspondre au point $(t,x) \in B+S'$ un point $(t+\tau,y,\tau) \in S' \times \langle 0,+\infty \rangle$ tel que $(t+\tau,y)$ est le conséquent ([10] p. 291, [4]) de P. L'ensemble $B+S'$ se transforme ainsi en un ensemble homéomorphe $\widetilde{B} \subset S' \times \langle 0,+\infty \rangle$ de points (u,y,τ) satisfaisant aux conditions

$$(u,y) \in S' \quad \text{et} \quad 0 \leqslant \tau < F(u,y),$$

où $F(u,y)$ est une fonction semi-continue inférieurement. En appliquant le théorème de Baire sur l'approximation d'une telle fonction par une suite croissante de fonctions continues, on démontre sans peine que l'ensemble $\widetilde{B}$ est homéomorphe à $S' \times I$.

Le théorème II ressemble au théorème 4 ([11], p. 147). Les autres théorèmes énoncés par T. Ważewski dans [11] pourraient aussi être adaptés au cas où la condition d'unicité ne subsisterait plus; cela exigerait pourtant une modification convenable du théorème II.

THÉORÈME III. *Supposons que les hypothèses des théorèmes précédents soient encore remplies et que les ensembles Ω et ω soient représentés respectivement par les inégalités*

$$(\Omega) \qquad a < t,$$
$$(\omega) \qquad a < t < 0,$$

et soit Π le plant $= 0$. Alors les ensembles $\omega+\Pi-U$ et $(\Pi-U)\times I$ sont homéomorphes et cette homéomorphie fait correspondre $(0,x,0)$ à $(0,x) \subset \Pi-U$.

COROLLAIRES. *Supposons que les fonctions $f_\nu(t,x)$ soient bornées. Alors l'ensemble U peut être considéré comme une zone d'émission négative d'un ensemble $V \subset \omega+\Pi$, fermé par rapport à $\omega+\Pi$* [3]. *On en déduit facilement p. ex. le théorème bien connu de Kneser* [7]. *On peut aussi utiliser le théorème III dans la discussion d'exemples analogues à ceux des travaux* [1], [8] *et* [10]. *Le théorème II donne immédiatement une généralisation d'un théorème de J. Szarski* [9] *sur les ensembles des intégrales asymptotiques. D'autres applications moins simples feront l'objet d'une communication ultérieure.*

REMARQUE. Tous les résultats établis dans cette note peuvent être généralisés au cas des équations au paratingent ([13], [14], [2]).

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MATHÉMATIQUE

Sur le mouvement des corps friables

par

W. WOLIBNER

Présenté par H. STEINHAUS le 25 Mai 1956

Lorsqu'on envisage le mouvement d'une grande quantité de corps solides, p. ex. du sable, il semble naturel de remplacer cette grande quantité de petits corps par un milieu continu. Cependant les forces superficielles dans ce milieu continu présenteraient un tenseur (car dans un tel milieu la pression doit toujours présenter un tenseur) dont les composantes ne seraient pas des fonctions linéaires de la vitesse de déformation, comme dans un liquide visqueux, mais conserveraient le caractère des réactions des corps solides. J'appelle un tel milieu continu *corps friable*, et — dans cette note — je m'occupe de sa mécanique.

Je dois cependant affirmer sous toutes réserves que je ne considère pas cette façon d'aborder le problème comme suffisamment justifiée théoriquement ou expérimentalement.

Soient: x_i les coordonnées du point du corps friable, $i=1,2,3$, ou $i=1,2$, selon que le corps est à trois ou à deux dimensions; ϱ la densité du corps; F_i les composantes des forces, rapportées à l'unité de masse; v_i les composantes de la vitesse du corps; p_{ij} le tenseur de la pression; α_i les cosinus directeurs de la normale de l'élément superficiel σ, et enfin p_n et p_s les composantes normales et tangentielles de la pression exercée sur l'élément σ. J'admets que v_i sont continuellement deux fois dérivables par rapport à x_i, et continuellement une fois dérivables par rapport à t, et que p_{ij} sont continuellement une fois dérivables par rapport à x_i.

Selon l'hypothèse fondamentale admise ci-dessus, il faut distinguer

I. le cas du frottement de glissement à l'état de mouvement,

II. le cas du frottement de glissement à l'état de repos.

Dans le premier cas on aurait pour un élément superficiel arbitraire

$$|p_s| = \mu |p_n|, \tag{1}$$

où μ désigne le coefficient constant du frottement. Lorsque dans le point en question la pression ne s'annule pas pour toutes les directions α_i, il

existe une quadrique directrice $\varphi(a_i)$; il résulte de l'équation (1) que

$$\frac{|\varphi(a_i)|}{\operatorname{grad}\varphi(a_i)} = \text{const.}, \tag{2}$$

lorsque pour les directions a_i le cas I se présente.

Quand les directions a_i, correspondant au cas I, forment sur la surface de la sphère unitaire un domaine à deux dimensions, il résulte de (2) que la quadrique directrice est une sphère, et il s'ensuit de (1) que $\mu = 0$. Or, lorsque dans le point considéré le tenseur de la pression ne s'annule pas et $\mu \neq 0$, le cas I ne peut pas avoir lieu pour les directions formant sur la surface de la sphère unitaire un ensemble partout dense. De là, par la continuité des conditions du mouvement correspondant au cas II — que je présenterai dans la suite — il résulte que les conditions en question doivent être remplies pour toutes les directions. Ainsi le cas I ne peut jamais avoir lieu.

Dans le cas II nous obtiendrons, au lieu de (1), l'inégalité

$$|p_s| \leqslant \mu |p_n|. \tag{3}$$

On a

$$p_n \geqslant 0, \tag{4}$$

puisque la pression normale ne peut évidemment pas être négative dans un corps friable.

En établissant les conditions du mouvement du corps friable dans l'entourage du point correspondant au cas II, on peut évidemment négliger le mouvement du corps rigide et ne se borner qu'à la vitesse locale de la déformation u_i qui, dans le point $x_i + \Delta x_i$, s'exprime par

$$u_i = \sum_j h_{ij} \Delta x_j, \qquad h_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{5}$$

Par suite, il semble naturel de n'envisager que la composante tangentielle de u_i par rapport à l'élément σ, c'est-à-dire le vecteur dont les composantes sont

$$u_i - \alpha_i u_n = u_i - \alpha_i \sum_j \alpha_j u_j.$$

Par analogie aux lois du frottement des corps solides je considère

$$\frac{\partial(u_i - \alpha_i u_n)}{\partial_n} = 0$$

comme condition pour que le cas II ait lieu dans le point x_i pour l'élément σ.

Cette équation devant être satisfaite pour toutes les directions α_i, il résulte, en vertu de (5),

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} = \frac{\partial v_3}{\partial x_3}; \qquad \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = 0, \qquad i \neq j.$$

Ces conditions expriment que l'ellipsoïde des dilatations est une sphère.

Soit M le plus grand ensemble ouvert dans l'espace x_i, t dans lequel le tenseur de la pression est nul.

Examinons tous les cas du mouvement du corps friable, comme suit:

1. $\mu = 0$; le corps friable est un liquide parfait.

2. Dans l'ensemble ouvert M le mouvement est tel comme si chaque point du corps était un point matériel sur lequel n'agiraient que les forces rapportées à l'unité de masse.

J'admets pour les cas suivants que $\mu \neq 0$, et que l'on envisage le mouvement à l'extérieur de M.

3. Si le corps friable est incompressible, il résulte de (6) que le mouvement de ce corps est un mouvement de corps rigide.

4. Si le corps friable est à deux dimensions et compressible (cas peu important au point de vue pratique), il résulte de (6) que $v_1 + iv_2$ est une fonction analytique de la variable $x_1 + ix_2$.

5. Si le corps friable est à trois dimensions et compressible, il s'ensuit de (6), par un calcul élémentaire, que son mouvement ne se compose que d'un mouvement de corps rigide, d'une dilatation homogène et d'un mouvement dont les composantes de la vitesse v_i' s'expriment comme suit:

$$v_i' = a_i(-x_i^2 + x_{i+1}^2 + x_{i+2}^2) - 2a_{i+1}x_i x_{i+1} - 2a_{i+2}x_i x_{i+2},$$

où a_i ne dépend pas de x_i.

Il ne résulte des considérations présentées ci-dessus aucune relation entre le mouvement du corps friable et le temps.

Pour que le mouvement en question puisse avoir lieu, il faut que l'équation de continuité, l'équation caractéristique et les trois équations du mouvement

$$\varrho\left(F_i - \frac{dv_i}{dt}\right) = \frac{\partial p i_1}{\partial x_1} + \frac{\partial p i_2}{\partial x_2} + \frac{\partial p i_3}{\partial x_3}, \qquad p_{ij} = p_{ji}, \tag{7}$$

soient satisfaites.

Dans le 3-me cas, le mouvement de corps rigide peut être absolument arbitraire, puisqu'on peut poser $p_{ij} = 0$, lorsque $i \neq j$ et déterminer les p_{ii} par les équations (7). Aux p_{ii} ainsi obtenus on peut ajouter la même constante suffisamment grande afin que les inégalités (3) et (4) soient satisfaites.

Dans les cas 4 et 5 la pression normale (moyenne) est déterminée au moyen de la densité par le mouvement du corps friable à l'aide de l'équation de continuité et de l'équation caractéristique; ainsi le mouvement en question ne peut être réalisé que s'il existe six fonctions p_{ij} satisfaisant aux trois équations (7) et aux inégalités (3) et (4).

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MATHÉMATIQUE

Sur les décompositions en trois continus

par

A. LELEK

Présenté par K. KURATOWSKI le 19 juin 1956

Cette communication a été suggérée par deux questions. L'une, de H. Steinhaus, concernait la propriété suivante (cf. [5], p. 125) de la sphère (considérée ici comme un continu X):

(S) *X étant décomposé en trois continus, $X=K_1+K_2+K_3$, tels que $K_m K_n \neq 0$ pour $m,n=1,2,3$, on a $K_1K_2K_3\neq 0$.*

L'autre, de B. Knaster, était de savoir si (S) équivaut à *l'unicohérence*, c'est-à-dire à la propriété suivante (que les sphères à plus d'une dimension possèdent notoirement):

(U) *X étant décomposé en deux continus, $X=C_1+C_2$, leur partie commune C_1C_2 est un continu.*

La réponse est que les propriétés (S) et (U) sont équivalentes pour tous les X localement connexes (images continues du segment rectiligne) et que, pour les autres X, (U) entraîne (S), mais (S) peut se présenter sans (U). C'est donc, en particulier, une réponse aux deux questions (vue la connexité locale des sphères à un nombre quelconque de dimensions).

THÉORÈME 1. (U) *entraîne* (S).

Démonstration. En posant $K_1=C_1$ et $K_2+K_3=C_2$, on a $X=C_1+C_2$ et il résulte de l'hypothèse $K_2K_3\neq 0$ que C_2 est aussi un continu. $C_1C_2=K_1K_2+K_1K_3$ étant un continu en vertu de (U) et ses deux sommandes étant compactes, leur partie commune $(K_1K_2)(K_1K_3)=K_1K_2K_3$ ne peut être vide. On a donc (S).

THÉORÈME 2. (S) *entraîne* (U) *pour les X localement connexes.*

Démonstration. D'après un théorème de Kuratowski (voir [1], p. 148 et 149), la propriété suivante, établie par L. E. J. Brouwer pour le plan, entraîne (U) pour les X localement connexes:

(B) *R étant une région et $X-R$ un continu,* $\mathrm{Fr}(R)$ *est un continu.*

Il suffit donc de déduire (B) de (S).

X étant un continu localement connexe et $X-R$ étant un continu, R se laisse décomposer comme suit ([2], 2, p. 189):

$$R=\sum_{i=1}^{\infty} H_i \text{ , où } H_i \text{ sont des continus,} \tag{1}$$

$$H_i \subset \operatorname{Int}(H_{i+1}), \tag{2}$$

$$X-H_i \text{ sont connexes pour } i=1,2,\ldots \tag{3}$$

Il résulte de (1) et (2) que ([3], (5), p. 586)

$$\operatorname{Fr}(R)=\prod_{i=1}^{\infty}\overline{R-H_i}. \tag{4}$$

Supposons, contrairement à (B), que $\operatorname{Fr}(R)$ ne soit pas un continu. Alors, au moins un des facteurs $\overline{R-H_i}$ ne l'est pas non plus (théorème de G. Cantor). Omettons-en désormais l'indice i et déduisons quelques formules. Il vient d'être supposé que

$$\overline{R-H}=M+N, \quad M=\bar{M}\neq 0\neq N=\bar{N}, \quad MN=0, \tag{5}$$

d'où, $\bar{R}$ étant un continu,

$$H\neq 0. \tag{6}$$

En appliquant (5), on a $X=(X-R)+R=(X-R)+(R-H)+H\subset(X-R)+\overline{R-H}+H=(X-R)+M+N+H$, c'est-à-dire

$$X=(X-R)+(M+H)+(N+H). \tag{7}$$

Envisageons les parties communes de ces sommandes. On a $X-H=(X-R)+(R-H)=(X-R)+\overline{R-H}=(X-R)+M+N$ d'après (1) et (5), les ensembles $(X-R)+N$ et M étant fermés. On a donc $[(X-R)+N]M\neq 0$ en vertu de (3) et comme $MN=0$ d'après (5), il vient $(X-R)M\neq 0$. Le rôle de M et N étant symétrique, il vient de même $(X-R)N\neq 0$. À plus forte raison

$$(X-R)(M+H)\neq 0\neq(X-R)(N+H). \tag{8}$$

Enfin, l'inclusion triviale $R-H\subset R$ entraîne $\overline{R-H}-H\subset\bar{R}-H$; réciproquement, H étant un continu d'après (1), on a $H=\bar{H}$, d'où $\bar{R}-H=\bar{R}-\bar{H}\subset R-H$ et, en soustrayant H des deux membres, $\bar{R}-H\subset\overline{R-H}-H$. Les deux inclusions donnent l'égalité $\overline{R-H}-H=\bar{R}-H$, dont il résulte en vertu de (5) que

$$\bar{R}-H=(M-H)+(N-H) \quad \text{et} \quad (\overline{M-H})(\overline{N-H})\subset MN=0. \tag{9}$$

Vu que $\bar{R}$ et H sont des continus et que $H\subset\bar{R}$ en vertu de (1), on conclut de (9) que

(10) les ensembles $(M-H)+H=M+H$ et $(N-H)+H=N+H$ sont des continus

([2], 3, p. 108) et il vient en vertu de (6),

$$(M+H)(N+H)=MN+H\neq 0. \tag{11}$$

Posons $K_1=X-R$, $K_2=M+H$ et $K_3=N+H$. Ce sont des continus: le premier par l'hypothèse sur $X-R$ et les deux autres d'après (10). On a $X=K_1+K_2+K_3$ en vertu de (7). On a $K_mK_n\neq 0$ pour $m,n=1,2,3$ en vertu de (8) et (11). Cependant, en vertu de (5) et (1),

$$K_1K_2K_3=(X-R)(M+H)(N+H)=(X-R)(MN+H)=(X-R)H=0,$$

contrairement à (S).

EXEMPLE. Soit X le continu composé de la circonférence $x^2+y^2=1$ et de deux courbes S_g et S_d disjointes, se condensant extérieurement sur ses moitiés gauche et droite respectivement et définies par l'équation commune

$$y=\sin\frac{\pi}{\varrho-1} \quad \text{où} \quad \varrho=\sqrt{x^2+y^2} \quad \text{et} \quad 1<\varrho\leqslant 2$$

(cf. [4], fig. 3, p. 22). X n'est pas localement connexe, à savoir aux points de la circonférence, et *n'est pas unicohérent*: en posant $C_1=\overline{S_g}$ et $C_2=\overline{S_d}$, on a en effet $X=C_1+C_2$, où les deux sommandes sont des continus, tandis que C_1C_2 ne l'est pas, étant composé de deux points, à savoir de $(0,-1)$ et $(0,1)$.

Cependant, *X a la propriété* (S).

En effet, soit $X=K_1+K_2+K_3$ une décomposition de X en trois continus empiétant chacun sur chacun. Soit K_1 celui qui contient le point $(-2,0)$. *Le continu X étant irréductible entre les points* $(-2,0)$ *et* $(2,0)$, si K_1 contient aussi le point $(2,0)$, on a $K_1=X$, d'où $K_2K_3\subset K_1$ et par conséquent $K_1K_2K_3=K_2K_3\neq 0$. Sinon, soit K_2 celui des trois continus qui contient le point $(2,0)$. On a alors, pour la même raison, $K_1+K_2=X$, d'où $K_3\subset K_1+K_2$, c'est-à-dire $K_3=(K_1+K_2)K_3=K_1K_3+K_2K_3$, ce qui est une décomposition du continu K_3 en deux ensembles fermés non-vides. Par conséquent $(K_1K_3)(K_2K_3)=K_1K_2K_3\neq 0$. Ainsi, la thèse de (S) est réalisée dans deux cas.

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Polarisation of Nucleons from Photo-disintegration of Deuteron. I. Medium Energies

by

W. CZYŻ and J. SAWICKI

Presented by L. INFELD on May 9, 1956

No experimental data concerning polarisation of nucleons from photo-disintegration of deuterium are at present available, and no theoretical study of the problem has so far been published. Since polarisation is, in general, an effect sensitive to the choice of nucleon-nucleon interaction, it seems that the investigation of polarisation in the photo-process on deuterium might be helpful in obtaining additional information concerning the type of n-p interaction — e. g. the measurement of the sign of polarisation and comparison with the theory might rule out some types of interaction.

The aim of the present paper is to calculate the polarisation of nucleons from the deuteron photo-effect on the method analogous to that of our previous note [1]. The n-p system is assumed to interact through a potential containing a tensor term. Then J, M_J, S, π are constants of the motion (π is the parity as defined in [2]). The energy eigenfunctions $\psi_E(J, M_J, \pi)$ are then simultaneous eigenfunctions of these operators and $\Phi_E(L, M_L)$correspond to singlet states. Thus, the wave function of the n-p system may be represented as a superposition of these functions:

$$\psi(\vec{r},t) = \Big(\sum_{E'} + \int dE'\Big)\Big\{\sum_{L,M_L} b_{E'}(L, M_L)\Phi_{E'}(L, M_L) + \\ + \sum_{J,M_J\pi} a_{E'}(J, M_J, \pi)\psi_{E'}(J, M_J, \pi)\Big\}, \tag{1}$$

where

$$\psi_{E'}(J, M_J, \pi = +1) = R_{J,J}(E')\, y^{M_J}_{J,J,1}\, e^{-i\frac{E't}{\hbar}}, \\ \psi_{E'}(J, M_J, \pi = -1) = \{R_{J,J-1}(E')\, y^{M_J}_{J,J-1,1} + R_{J,J+1}(E')\, y^{M_J}_{J,J+1,1}\}\, e^{-i\frac{E't}{\hbar}}. \tag{2}$$

$R_{JL}(E')$ depends on the n-p distance r, and $y^{M_J}_{JL1}$ is the eigenfunction of the set of the commuting operators J, M_J, S, π and is composed of the spherical harmonics $Y_{LM_L}(\vartheta,\varphi)$ and the triplet spin functions X_1, X_0, X_{-1}.

In the present paper, we shall confine ourselves to the dipole electric transitions and estimate the quadrupole electric corrections. The dipole magnetic transitions are taken only to the ${}^1S_0$ state. It is seen from (2) that stationary states of different L are coupled for one J by the tensor forces. For the dipole electric transitions from the ${}^3S_1+{}^3D_1$ ground state, we have the following final states: ${}^3P_0, {}^3P_1, {}^3P_2+{}^3F_2$. However, the ${}^3F_2$ state admixture is so small as to be negligible (see [3]-[4]). Thus, a similar situation in the calculations is obtained as in the case of pure $\vec{L}\vec{S}$ coupling (there are ${}^3P_0$ and ${}^1S_0$ final states only).

Similarly as in [1] we obtain:

$$\psi(\vec{r},t) \sim f\chi r^{-1} e^{ikr - i\frac{Et}{\hbar}}, \tag{3}$$

where the reaction amplitude $f\chi = \sum_{m_s=-1}^{1} f_{m_s}\chi_{m_s} + g\chi^0_0$. In our case we have

$$f\chi = e^{i\delta_S} y^0_{000} H^{10,0}_{1m_J} - i\left\{ e^{i\eta_0} y^0_{011} H'^{00}_{1mj} + e^{i\eta_1} \sum_{M_J=-1}^{1} y^{M_J}_{111} H'^{1M_J}_{1mj} + e^{i\eta_2} \sum_{M_J=-2}^{2} y^{M_j}_{211} H'^{2M_J}_{1mj} \right\} \tag{4}$$

where η_J are the ${}^3P_J$-states and δ_S is the ${}^1S_0$-state phaseshifts, H'^{J,M_J}_{j,m_j} are the matrix elements of the dipole electric transitions from the ${}^3S_1+{}^3D_1$ ground state to the ${}^3P_J$ final states. Similarly as in [1], the quantisation axis is chosen along the vector $\vec{k}\times\vec{k}_0$, where $\vec{k}$ is the wave vector of the n-p relative motion, and $\vec{k}_0$ is the wave vector of the photon. H' is proportional to $\vec{e}\vec{r} + \frac{\hbar}{2Mc}(\mu_n - \mu_p)\left(\frac{\vec{k}_0}{k_0}\times\vec{e}\right)(\vec{\sigma}_n - \vec{\sigma}_p)$, where $\vec{e}$ is the polarisation vector of the photon. In our frame of reference (see [1]) we have for the last expression

$$\left[z + \frac{\hbar}{2Mc}(\mu_n - \mu_p)(\sigma_{nx} - \sigma_{px})\right]\cos\Phi + \left[x - \frac{\hbar}{2Mc}(\mu_n - \mu_p)(\sigma_{nz} - \sigma_{pz})\right]\sin\Phi, \tag{5}$$

where $\Phi = \sphericalangle(\vec{e}, \vec{k}\times\vec{k}_0)$.

Thus we may write: $f_{m_j} = f^z_{m_s}\cos\Phi + f^x_{m_s}\sin\Phi$ and $g = g^z\cos\Phi + g^x\sin\Phi$.

With the help of the reaction amplitude (4) the polarisation vector $\vec{P}$ may be written in the form:

$$\vec{p} = \frac{\{\langle f\chi|\vec{\sigma}_n|f\chi\rangle\}_{av}}{\{\langle f\chi|f\chi\rangle\}_{av}}, \tag{6}$$

where $\{\ \}_{av}$ denotes averaging over the direction of photon polarisation and magnetic quantum numbers m_j of the initial state. In our frame of reference, the components P_x and P_y vanish. In analogy with [1] we

obtain the degree of polarisation:

$$P=P_2=\frac{\sum_{mj}(|f^x_{+1}|^2-|f^x_{-1}|^2+2Re(g^{x*}f^x_0)+2Re(g^{z*}f^z_0))}{\sum_{m_s}\sum_{m_j}(|f^z_{m_s}|^2+|f^x_{m_s}|^2+|g^x|^2+|g^z|^2)}. \tag{7}$$

As a final result we obtain

$$P=\frac{1}{2\bar{\sigma}}\Big\{[\tfrac{1}{3}I_0I_2\sin(\eta_0-\eta_2)+\tfrac{1}{2}I_1I_2\sin(\eta_1-\eta_2)]\sin 2\theta+ \\ +\frac{2\hbar}{Mc}(\mu_p-\mu_n)I_SI_1\sin(\eta_1-\delta_S)\sin\theta\Big\} \tag{8}$$

$$\bar{\sigma}=\tfrac{1}{9}I_0^2+\tfrac{5}{1\,6}I_1^2+\tfrac{7}{1\,4\,4}I_2^2-\tfrac{1}{1\,8}I_0I_2\cos(\eta_0-\eta_2)-\tfrac{1}{8}I_1I_2\cos(\eta_1-\eta_2)- \\ -\cos 2\theta[\tfrac{1}{1\,6}I_1^2+\tfrac{7}{4\,8}I_2^2+\tfrac{1}{6}I_0I_2\cos(\eta_0-\eta_2)+\tfrac{3}{8}I_1I_2\cos(\eta_1-\eta_2)]+ \\ +\frac{\hbar^2}{M^2c^2}(\mu_n-\mu_p)^2I_S^2, \tag{9}$$

where in the notation of [4]:

$$I_0=\int_0^\infty r\,v_0(u-2^{-\frac{1}{2}}w)\,dr,$$

$$I_1=\int_0^\infty r\,v_1(u+2^{-\frac{1}{2}}w)\,dr,$$

$$I_2=\int_0^\infty r\,v_2(u-\tfrac{1}{2}2^{-\frac{1}{2}}w)\,dr,$$

$$I_S=\int_0^\infty u_0u\,dr,$$

where $v_J=rR_{J1}(E)$; $u=rR_{10}(-\epsilon)$ and $w=rR_{12}(-\epsilon)$ are the radial wave functions of the deuteron ground state with the binding energy $=\epsilon$, and θ is the c. m. system scattering angle. The expression (9) gives the angular distribution of photonucleons in exact accordance with that of Rarita and Schwinger [4].

The n-p system interaction for the energies involved here was assumed in the form given in [4], i. e. of the "weak tensor forces" type with the square well radial dependence. Similarly as in [4], three types of exchange forces were considered: "symmetrical", "charged" and "neutral". Numerical values of the interaction parameters were assumed as in [4] and [5], i. e. in the notation of [4] and [5]: $\gamma=0.775$. $V_0=13.89$ MeV, $r_0=2.80\cdot10^{-13}$ cm. The integrals I_J and I_S and the phaseshifts η_J and δ_S were exactly calculated for $\hbar\omega=17.6$ MeV. As in [1] for the electric dipole approximation, the angular dependence of the polarisation is obtained in the form:

$$P=\frac{a\sin 2\theta+d\sin\theta}{b+c\cos 2\theta}. \tag{10}$$

Numerical results are as follows. The values of $P_{\max}$ for the three types of interaction are given in Table I.

TABLE I

	Symmetrical Theory	Charged Theory	Neutral Theory
$P_{\max}$	−20.5%	−13.9%	6.3%

On introducing the quadrupole interaction, it is necessary to replace the expression $\vec{e}\cdot\vec{r}$ by:

$$(11)\qquad \vec{e}\vec{r}+i\mu(\vec{e}\vec{r})\left(\frac{\vec{k}_0}{k_0}\vec{r}\right)=(z+i\mu zy)\cos\Phi+(x+i\mu xy)\sin\Phi,$$

where $\mu=\frac{1}{4}\frac{\omega}{c}$. Since the "weak tensor forces" produce, at $\hbar\omega=17.6$ MeV, small phaseshifts in the ${}^3D_J$ states, we may for the sake of simplicity, assume the ${}^3D_J$ final states as free (no splitting results). The small admixture of the ${}^3D_1$ state in the deuteron ground state is neglected in the quadrupole transitions. Under these simplifications, on performing calculations in a manner similar to that adopted previously, we obtain:

$$(12)\qquad \begin{aligned} P_q=\tfrac{1}{2}\Big\{&\Big[\tfrac{1}{3}I_0I_2\sin(\eta_0-\eta_2)+\tfrac{1}{2}I_1I_2\sin(\eta_1-\eta_2)-\frac{\mu}{6}I_D\cos\theta(5I_2\sin\eta_2-\\ &-2I_0\sin\eta_0-3I_1\sin\eta_1)]\sin 2\theta+\frac{2\hbar}{Mc}(\mu_n-\mu_p)I_S\times[I_1\sin(\delta_S-\eta_1)\sin\theta+\\ &+\tfrac{1}{2}\mu I_D\sin\delta_S\sin 2\theta]\Big\}\{\sigma+\tfrac{1}{2}[\tfrac{3}{7}\mu^2I_D^2\sin^2 2\theta+\mu I_D\sin\theta\sin 2\theta(\tfrac{1}{3}I_0\cos\eta_0+\\ &+I_1\cos\eta_1+\tfrac{5}{3}I_2\cos\eta_2)]\}^{-1}, \end{aligned}$$

where $I_D=\int_0^\infty r^4R_D^*R_{10}dr$ and R_D is the D state radial wave function. Now we see that the contribution of the quadrupole-dipole electric interference to the numerator of the polarisation vanishes when the ${}^3P$ state splitting is zero – i. e. in the case of neglecting the level splitting in the final states.

The angular dependence of the polarisation obtained in (12) is of the type:

$$(13)\qquad P_q=\frac{(a+a'\cos\theta)\sin 2\theta+d\sin\theta+d'\sin 2\theta}{b+c\cos 2\theta+b'\sin^2 2\theta+c'\sin\theta\sin 2\theta}.$$

The corrections coming from the quadrupole transitions – i. e. from a', b', c', d' – were estimated and found to be not very important.

Our thanks are due to Professor L. Infeld and Professor H. Niewodniczański for their kind interest in this work. We are indebted to

Dr. L. Rozenzweig for providing us with his paper prior to publication [6]. This paper called our attention to the importance of magnetic transitions erroneously omitted in our previous manuscript.

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CHEMISTRY

On Formation of a Derivative of 1, 5-diazabicycle (3, 3, 3) undekane from 1-nitropropane, Formaldehyde and Ammonia

by

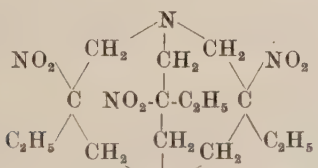
J. K. N. JONES, R. KOLIŃSKI, H. PIOTROWSKA and T. URBAŃSKI

Communicated by T. URBAŃSKI at the meeting of March 19, 1956

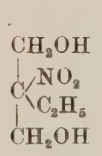
It was formerly found [1] that 1-nitropropane reacting with an excess of formaldehyde and ammonia could furnish derivatives of tetrahydro-1, 3-oxazine and a derivative of 1-oxa-3-azacyclooctane.

The formation of derivatives of tetrahydro-1, 3-oxazine has also been observed in the case of other nitroparaffins [2]—[5].

It has now been found that 1-nitropropane, when reacting with formaldehyde and excess of ammonia, can furnish *3, 7, 10-trinitro-3, 7, 10-triethyl-1,5-diazabicyclo (3, 3, 3) undekane* (*I*) (m. p. 107-108°). The compound was formed with a relatively high yield (c. 30%) when 2-nitro--2-ethylpropandiol-1,3 (II) produced from 1-nitropropane and two mols of formaldehyde reacted with aqueous solution of five mols of ammonia at room temperature (c. 25°). A resinous product was originally formed, and this crystallised on standing, yielding compound (I).



(I)



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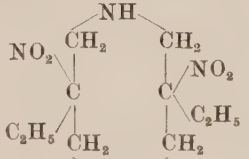


(III)

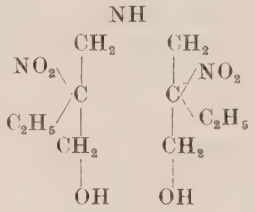
The substance (I) is a weak tertiary base, which yielded *monohydrochloride* (m. p. 143-145°) but no picrate. When warmed with conc. hydrochloric acid or boiled with hydrochloric acid diluted with ethyl alcohol, formaldehyde and 1-nitropropane was split off and *3, 7-dinitro-3, 7-diethyl-1, 5-diazacyclo-octane* (III) (m. p. 63-64°) resulted. Product (III) has properties much similar to the derivatives of 1, 5-diazacyclooctane which have been formerly prepared from nitroethane [2] and 1-nitrobutane [3]: it formed a monohydrochloride (m. p. 172-173°) with nitrous acid giving

1-nitrosoderivative (m. p. 101-103°), with tosyl chloride, 1-tosyl derivative (m. p. 138-140°).

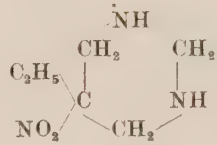
Compound (III) when warmed with 2-nitro-2-ethylpropandiol-1,3 (II) again yielded (I). Compound (III) was also formed when 2-nitro-2-ethylpropandiol (II) was heated to c. 100° with an excess of ammonia.

Both cyclic amines — (I) and (III) — have been also formed, when bis-(2-nitro-2-hydroxymethylbutyl)-amine (IV) [1], reacted with an excess (5 mols) of ammonia. The reaction at room temperature yielded product (I); at 100° — the amine (III) as a main product and a certain quantity of the amine (I) were formed.

From the aqueous layer formed upon the action on 2-nitro-2-ethylpropandiol-1,3 (II) of aqueous ammonia, another cyclic amine, *5-nitro--5-ethylhexahydropyrimidine* (V) with the yield of c. 4% was also isolated in the form of *dihydrochloride* (m. p. 159-160°). This was analogous to the compound previously described [2].



(IV)



(V)

Experimental details will be reported elsewhere [7].

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CHEMISTRY

Compositions of Ternary Positive Azeotropes. II

by

W. MALESIŃSKI

Presented by W. ŚWIĘTOSŁAWSKI on May 10, 1956

1. Introductory Remarks

In a previous paper [1] the following system of equations was derived, to determine the parameters of azeotropic points for ternary positive azeotropes.

$$\Delta S_1(T_{Az}-T_1)+A_{12}(1-x_1)^2+A_{23}x_3^2=0, \tag{1a}$$

$$\Delta S_2(T_{Az}-T_2)+A_{12}x_1^2+A_{23}x_3^2=0, \tag{1b}$$

$$\Delta S_3(T_{Az}-T_3)+A_{12}x_1^2+A_{23}(1-x_3)^2=0. \tag{1c}$$

In these equations, $\Delta S_1, \Delta S_2, \Delta S_3$ stand for the entropies of evaporation at the respective boiling temperatures, T_1, T_2, T_3, of pure components 1, 2, 3, under constant pressure P; T_{Az} is the boiling temperature of the ternary azeotrope, and A_{12}, A_{23} are the regular solution constants of binary mixtures (1,2) and (2,3), x_1 and x_3 being the mole fractions of components 1 and 3 in the ternary azeotropic mixture.

If all entropies of evaporation are assumed to be equal to one another, then

$$\Delta S_1=\Delta S_2=\Delta S_3,$$

and the composition of ternary azeotrope (1,2,3) may be expressed in terms of compositions of binary azeotropes (1,2) and (2,3):

$$x_1^{(1,2,3)}=x_1^{(1,2)}, \tag{2a}$$

$$x_3^{(1,2,3)}=x_3^{(2,3)}. \tag{2b}$$

The geometrical interpretation of relations (2a), (2b) was called the "rule of parallel lines". Attention is drawn to the fact that component "2" should have a medium value of internal pressure. This is in conformity with Świętosławski's definition of the azeotropic agent [2]. In the present paper, an attempt is made to extend the previous consideration [1] to those systems in which the entropies of evaporation are not equal to one

another. These inequalities exert their influence on the composition of the ternary azeotrope, which produces deviations from the rule of parallel lines.

2. General solution

Substracting respectively Eqs. (1a) and (1c) from Eqs. (1b), we obtain:

$$\Delta S_1(T_{Az}^{(1,2,3)}-T_1)-\Delta S_2(T_{Az}^{(1,2,3)}-T_2)+A_{12}(1-2x_1^{(1,2,3)})=0, \tag{5}$$

$$\Delta S_3(T_{Az}^{(1,2,3)}-T_3)-\Delta S_2(T_{Az}^{(1,2,3)}-T_2)+A_{23}(1-2x_3^{(1,2,3)})=0. \tag{6}$$

Solving Eqs. (5), (6) for the mole fractions x_1 and x_3, we obtain:

$$x_1^{(1,2,3)}=\frac{1}{2}+\frac{\Delta H_2-\Delta H_1}{2A_{12}}+\frac{T_{Az}^{(1,2,3)}(\Delta S_1-\Delta S_2)}{2A_{12}}, \tag{7}$$

$$x_3^{(1,2,3)}=\frac{1}{2}+\frac{\Delta H_2-\Delta H_3}{2A_{23}}+\frac{T_{Az}^{(1,2,3)}(\Delta S_3-\Delta S_2)}{2A_{23}}, \tag{8}$$

$\Delta H_1, \Delta H_2, \Delta H_3$ being the latent heats of evaporation of the pure components, at their respective boiling temperatures under the chosen constant pressure P. The composition of the binary azeotropes may be found in a similar manner. The mole fractions $x_1^{(1,2)}$ and $x_3^{(2,3)}$ have been determined previously [3] as follows:

$$x_1^{(1,2)}=\frac{1}{2}+\frac{\Delta H_2-\Delta H_1}{2A_{12}}+\frac{T_{Az}^{(1,2)}(\Delta S_1-\Delta S_2)}{2A_{12}}, \tag{9}$$

$$x_3^{(2,3)}=\frac{1}{2}+\frac{\Delta H_2-\Delta H_3}{2A_{23}}+\frac{T_{Az}^{(1,2)}(\Delta S_3-\Delta S_2)}{2A_{23}}. \tag{10}$$

Using these two equations, we may rewrite Eqs. (7), and (8) in the following manner:

$$x_1^{(1,2,3)}=x_1^{(1,2)}+\frac{(T_{Az}^{(1,2)}-T_{Az}^{(1,2,3)})(\Delta S_2-\Delta S_1)}{2A_{12}}, \tag{11}$$

$$x_3^{(1,2,3)}=x_3^{(3,2)}+\frac{(T_{Az}^{(2,3)}-T_{Az}^{(1,2,3)})(\Delta S_2-\Delta S_3)}{2A_{23}}. \tag{12}$$

Expressing these equations in an abbreviated form, we have

$$x_1^{(1,2,3)}=x_1^{(1,2)}+\Delta x_1, \tag{13}$$

$$x_3^{(1,2,3)}=x_3^{(2,3)}+\Delta x_3, \tag{14}$$

where

$$\Delta x_1=\frac{(T_{Az}^{(1,2)}-T_{Az}^{(1,2,3)})(\Delta S_2-\Delta S_1)}{2A_{12}}, \tag{15}$$

$$\Delta x_3 = \frac{(T_{Az}^{(2,3)} - T_{Az}^{(1,2,3)})(\Delta S_2 - \Delta S_3)}{2A_{23}}. \tag{16}$$

Introducing into Eqs. (15) and (16) the values of the lower parts of the azeotropic ranges Z_d, which characterise the respective binary systems, we obtain:

$$\Delta x_1 = \frac{(T_{Az}^{(1,2)} - T_{Az}^{(1,2,3)})}{2Z_{d_{21}}}\left(1 - \frac{\Delta S_1}{\Delta S_2}\right), \tag{18}$$

$$\Delta x_3 = \frac{(T_{Az}^{(2,3)} - T_{Az}^{(1,2,3)})}{2Z_{d_{23}}}\left(1 - \frac{\Delta S_3}{\Delta S_2}\right). \tag{19}$$

It is easy to see that, if $\Delta S_1, \Delta S_2, \Delta S_3$ are equal to one another, Δx, and Δx_3 are equal to zero; for this reason, relations (13) and (14) are identical respectively to (2a) and (2b). If that equality is not fulfilled, the corrections Δx may be either positive or negative. If $\Delta S_1 > \Delta S_2$, then $\Delta x_1 < 0$, consequently $x_1^{(1,2,3)} < x_1^{(1,2)}$; if, however, $\Delta S_1 < \Delta S_2$, then $\Delta x_1 > 0$, and $x_1^{(1,2,3)} > x_1^{(1,2)}$. In the same way, the values ΔS_2 and ΔS_3 exert their influence on the sign of Δx_3. In general, eight cases may be distinguished, if we also take into account the fact that two entropies out of three may be equal to one another: $\Delta S_1 = -\Delta S_2 \neq \Delta S_3$, or $\Delta S_3 = \Delta S_2 \neq \Delta S_1$.

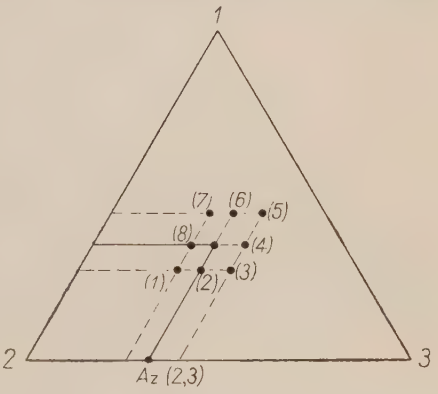

Fig. 1. Deviations from the parallel lines rule. The particular points represent the following cases: 1) $\Delta S_1 > \Delta S_2$, $\Delta S_3 < \Delta S_2$, 2) $\Delta S_1 > \Delta S_2$, $\Delta S_3 = \Delta S_2$. 3) $\Delta S_1 > \Delta S_2$, $\Delta S_3 < \Delta S_2$. 4) $\Delta S_1 = \Delta S_2$, $\Delta S_3 < \Delta S_2$, 5) $\Delta S_1 < \Delta S_2$, $\Delta S_3 < \Delta S_2$, 6) $\Delta S_1 < \Delta S_2$, $\Delta S_3 = \Delta S_2$, 7) $\Delta S_1 < \Delta S_2$, $\Delta S_3 > \Delta S_2$, 8) $\Delta S_1 = \Delta S_2$, $\Delta S_3 > \Delta S_2$.

The sings characterising the deviations may be predicted by the following rule. If the entropy of evaporation of the component 1 is larger than that of the main component 2, then the mole fraction of 1 should be smaller in the ternary than in the binary azeotrope (1,2). The same is true as regards the component 3. A reverse statement is also true: the decrease of ΔS_1, or ΔS_3, compared with ΔS_2, produces the increase of the respective mole franction x_1 or x_3 in the ternary azeotrope.

In order to calculate the values of Δx_1 and Δx_3, using Eqs. (18) and (19), we should know not only the values of $\Delta S_1, \Delta S_2, \Delta S_3$, characterising the pure components, and the parameters of the binary azeotropic points, but we should also know the boiling temperature of the ternary azeotrope, $T_{Az}^{(1,2,3)}$. Its value is not always available. For this reason, an attempt is made below, after admitting some approximations, to derive the relations which do not need a knowledge of the boiling temperature of the ternary azeotrope.

3. Approximate calculation of the corrections $\Delta x_1, \Delta x_3$,

Assuming all three entropies of evaporation to be equal to one another, an equation has been previously derived, relating the boiling temperature of the ternary azeotrope to the boiling temperatures, $T_{Az}^{(1,2)}, T_{Az}^{(2,3)}$, of the binary ones, and to that of the main component, T_2:

$$T_2 - T_{Az}^{(1,2,3)} = (T_2 - T_{Az}^{(1,2)}) + (T_2 - T_{Az}^{(2,3)}). \tag{20}$$

After rearranging the Eq. (20), we obtain:

$$T_{Az}^{(1,2)} - T_{Az}^{(1,2,3)} = T_2 - T_{Az}^{(2,3)} \equiv \delta_{23}, \tag{21}$$

$$T_{Az}^{(2,3)} - T_{Az}^{(1,2,3)} = T_2 - T_{Az}^{(1,2)} \equiv \delta_{21}, \tag{22}$$

δ_{12}, δ_{23} being the azeotropic boiling temperature depressions in relation to the main component 2. Introducing respectively Eqs. (21) and (22) into (18) and (19), we obtain:

$$\Delta x_1 = \frac{\delta_{23}}{2Z_{d_{21}}}\left(1 - \frac{\Delta S_1}{\Delta S_2}\right), \tag{23}$$

$$\Delta x_3 = \frac{\delta_{21}}{2Z_{d_{23}}}\left(1 - \frac{\Delta S_3}{\Delta S_2}\right). \tag{24}$$

In most cases, the error in values of Δx_1 and Δx_3, introduced by using Eqs. (21) and (22) for $T_{Az}^{(1,2\,3)}$, is relatively small, despite the fact that these equations are limited to the systems in which all the entropies of evaporation are equal to one another. The values of azeotropic compositions may be introduced into Eqs. (23) and (24) by the use of the equation

$$\delta_{21} = Z_{d_{21}}(x_1^{(1,2)})^2; \qquad \delta_{23} = Z_{d_{23}}(x_3^{(2,3)})^2, \tag{25}$$

which relates the boiling temperature depressions to the azeotropic compositions (3). Inserting Eqs. (25) into Eq. (23) and Eq. (24), we obtain;

$$\Delta x_1 = \frac{Z_{d_{23}}}{2Z_{d_{21}}}\left(1 - \frac{\Delta S_1}{\Delta S_2}\right)(x_3^{(2,3)})^2, \tag{26}$$

$$\Delta x_3 = \frac{Z_{d_{21}}}{2Z_{d_{23}}}\left(1 - \frac{\Delta S_3}{\Delta S_2}\right)(x_1^{(1,2)})^2. \tag{27}$$

After evaluating the corrections Δx_1 and Δx_3, we may also calculate the compositions of ternary azeotropes, in those systems in which the entropies of evaporation of components are not equal to one another.

4. Deviations from the parallel lines rule

Estimation of correction values for Δx_1 and Δx_3 leads to the conclusion that, in general, they may vary considerably. For example, if $\delta_{23} = 10°\text{C.}$, $2Z_{d_{21}} = 20°\text{C.}$, $\Delta S_1 = 20$ e. u., $\Delta S_2 = 25$ e. u., we obtain for the correction Δx_1 the value of 10 mole per cent, from Eq. (23). If, however, $\delta_{23} = 1°\text{C.}$, $2Z_{d_{21}} = 100°\text{C.}$, $\Delta S_1 = 21$ e. u., $\Delta S_2 = 20$ e. u., then Δx_1 equals -0.0005, i. e.

-0.05 mole %. In order to obtain a proper idea of the values of these corrections, it has to be taken into account that the increase in differences of ΔS is usually associated with the increase in the azeotropic range. Equations (23) and (24) indicate that Δx_1 and Δx_3 are proportional to the difference in ΔS, and inversely proportional to the value of the azeotropic range. Consequently, large differences in ΔS are associated with relatively small increases in Δx_1 and Δx_3 values. The increases both of ΔS and of azeotropic range produces decreases in Δx values.

Applying Eqs. (23) and (24), it is found that the fluctuations of Δx are of the order of several mole per cent. In some cases, however, they may exceed ten per cent.

Summary

1. The theoretical solution is given for determining the compositions of ternary azeotropes in those systems in which the entropies of evaporation of the components 1, 2, 3 are not equal to one another. The corrections Δx_1 and Δx_3 are to be introduced:

(13) $$x_1^{(1,2,3)} = x_1^{(1,2)} + \Delta x_1$$

(14) $$x_3^{(1,2,3)} = x_3^{(2,3)} + \Delta x_3$$

2. Equations are derived thus:

(18) $$\Delta x_1 = (T_{Az}^{(1,2)} - T_{Az}^{(1,2,2)}) \left(1 - \frac{\Delta S_1}{\Delta S_2}\right) (2Z_{d_{21}})^{-1}$$

(19) $$\Delta x_3 = (T^{(3,2)} - T_{Az}^{(1,2,3)}) (1 - \Delta S_3/\Delta S_2) (2Z_{d_{23}})^{-1},$$

making it possible to calculate the Δx_1 and Δx_3 values. In these equations, T_{Az} are the boiling temperatures of binary and ternary azeotropes; ΔS_1, ΔS_2, ΔS_3 stand for the entropies of evaporation of pure components, and $Z_{d_{21}}$, $Z_{d_{23}}$ — for the azeotropic ranges,

3. The corrections Δx_1 and Δx_3 are of the order of several mole per cent; in rare particular cases they may slightly exceed ten per cent.

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CHEMISTRY

The Potential of the Adsorption Micro-electrode in Air Contaminated by Reducing Gases

by

B. KAMIEŃSKI and J. KULAWIK

Communicated by B. KAMIEŃSKI on May 18, 1956

The adsorption micro-electrode described in former papers [1]-[8] was used as a sensitive detector of halogens in air [9]. The electrode is composed of a thin platinum wire fused into a glass rod. The ground cross-section of the rod discloses the end of the wire which serves as an electrode. A microscopic silica gel layer is put on the cross-section and forms the electrolyte as described in previous papers by B. Kamieński [7]. A suitable reference electrode (calomel electrode) completes the electric element. It provides an efficient apparatus for indicating slight traces of acid, basic, oxidising and reducing gases contaminating the atmosphere [3], [4], [8].

The influence of hydrogen on the potential of the electrode was examined. The potential changed when the air contained different quantities of hydrogen and surrounded the active end of the electrode. The first diagram represents the potential of the electrode (ordinate in millivolts) with time (minutes) when the composition (volume %) of the air changed from pure air to various mixtures – namely 0.18%, 1.8%, 18% and 100% hydrogen. Each change of the composition is marked by arrows in Fig. 1. It is seen that the potential changes very rapidly; the first arrow shows the influence of hydrogen (0.18%), the second the influence of fresh air and so on, until pure hydrogen and air were applied. Even slight traces of hydrogen, namely 0.18%, enforce a potential change of 175 mV. A tenfold increase in hydrogen content (1.8%) almost doubles the change in potential (318 mV). Hydrogen was removed by a stream of fresh air which changed the potential as may be seen in the Fig. 1 (second arrow). The potential of the electrode does not return exactly to the same position after the first application of hydrogen and air. Some

hydrogen molecules evidently diffuse into the platinum; however, when the process of saturating the electrode was repeated (1.8, 18, 100%) the potential of the electrode became fairly accurately reversible. The diffusion of hydrogen through the adsorption layer of silica gel is rapid, as may be seen from an examination of the rate of the change of potential with time. The potential change and the rate with time is much smaller when carbon monoxide is in the atmosphere, as may be seen in Fig. 2.

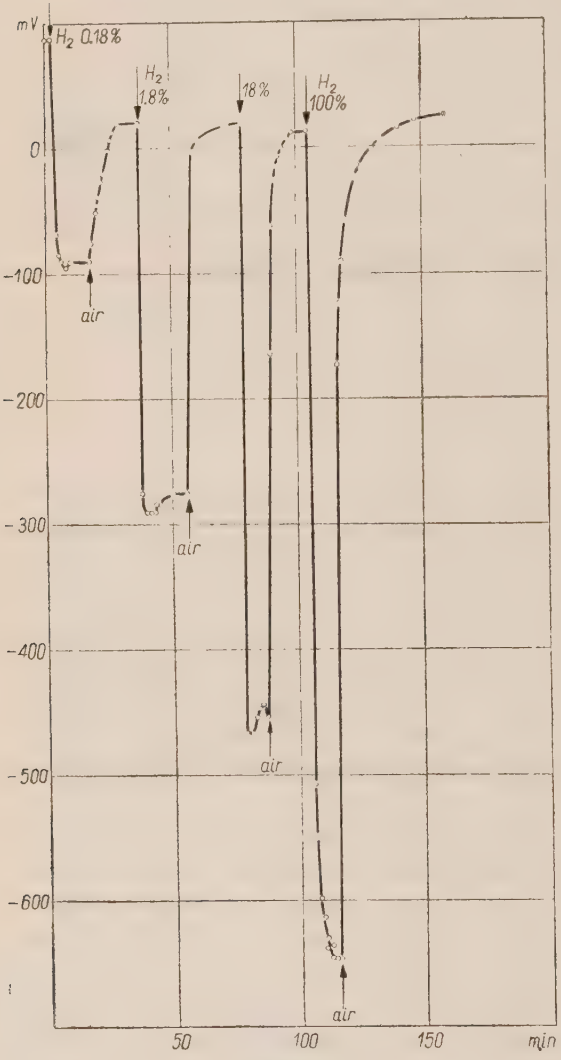

Fig. 1

Carbon monoxide was prepared by the action of sulphuric acid on formic acid, and suitably treated in adsorption bulbs and sintered glass filters for purification. Different mixtures gave different potential changes as shown in Fig. 2. Carbon monoxide is a poor reducing agent, and its action on the platinum electrode is also slight. The change of the potential amounts to c. 25 mV when the atmosphere contains 1% carbon monoxide. There is also some marked irreversible potential change of platinum under the influence of carbon monoxide; this seems to be a consequence of the affinity of the gas to platinum [13].

Hydrogen sulphide was the next reducing gas applied with a view to studying the behaviour of the platinum micro-electrode. According to reducing power, great potential changes were observed in the platinum electrode.

The change was c. 140 mV when the air contained 0.1% hydrogen sulfide, 210 mV when 1% of the hydrogen sulphide contaminated the

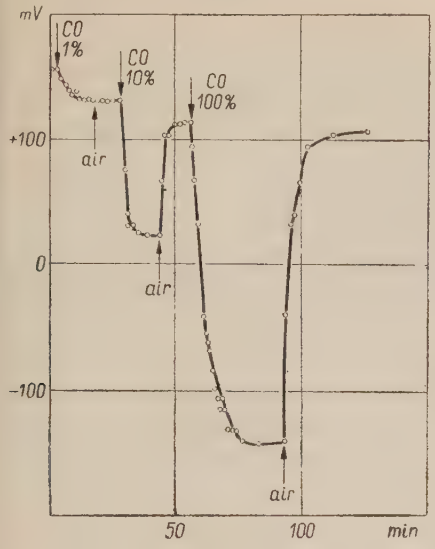

Fig. 2

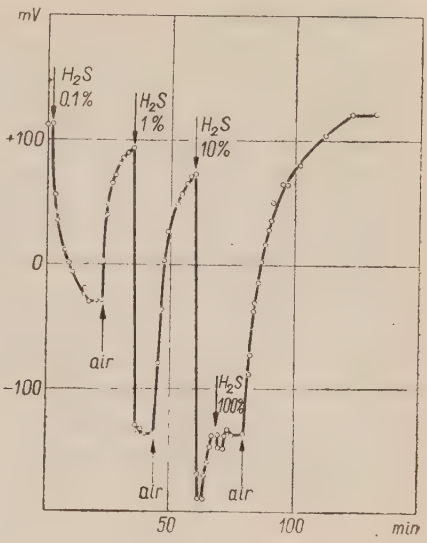

Fig. 3

atmosphere, and 265 mV when the air contained 10% of the gas. A further increase to 100% does not change the potential. Evidently, a saturation of the electrode with hydrogen sulphide follows when a certain volume percentage of this gas is present in air.

Even gases of indifferent nature (e. g. nitrogen) act indirectly on the electrode by removing oxygen from the air. Acetylene might act to some degree as a reducing gas and partially by removing a part of oxygen from the air. The potential of the electrode changes c. 300 mV when the atmosphere contains 10% acetylene. Pure acetylene changes the potential 500 mV as may be seen in Fig. 4. A fresh stream of air restore the potential with time asymptotically.

Summing up the results, it should be stated that the adsorption electrode makes it possible to detect not only traces of halogens but also traces of reducing gases in the air.

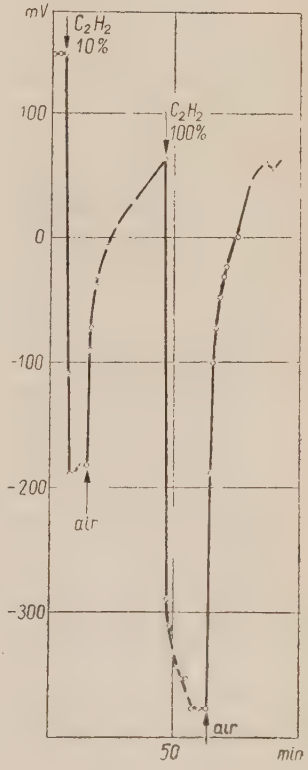

Fig. 4

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CHEMISTRY

The Influence of the Composition of ZnO-Fe_2O_3 and of ZnO-Cr_2O_3 Mixed Catalysts on Their Electric Conductivity and Catalytic Activity

by

A. BIELAŃSKI, J. DEREŃ, J. HABER, S. MROWEC

Presented by B. KAMIEŃSKI on May 11, 1956

In previous publications [1]-[3], the present authors stated that the catalytic reaction of the dehydrogenation of ethyl alcohol was accompanied by a change in the electric conductivity of the semiconducting oxides applied as catalysts. The character of the change depended upon the type of the semiconducting oxides. In the case when the catalyst was an *n*-semiconductor (ZnO), or a mixture of *n*-semiconducting oxides ($ZnO + Fe_2O_3$), the electric conductivity increased after the vapours of alcohol-water mixture were introduced up to a certain constant value — usually reached in the course of a few minutes. When the catalyst was a *p*-semiconducting oxide (Cr_2O_3), or a mixture of such oxides (MgO-Cr_2O_3), the conductivity decreased down to a constant value (region of lower temperatures), or at first decreased and then increased up to a constant value higher than the initial one (region of higher temperatures). If the catalyst was a mixture of *n*- and *p*-semiconducting oxides (ZnO-Cr_2O_3), an increase in the conductivity was observed for mixtures rich in *n*-conducting oxide and a decrease in the conductivity for mixtures rich in *p*-conducting oxide.

It was ascertained that in all cases there exists a linear dependence between the change observed, in the absolute value of the logarithm of the conductivity $|\Delta \log \sigma|$ and the yield of the catalytic reaction.

In the case of MgO-Cr_2O_3 mixed catalysts, it was shown that there also exists a certain relatively simple relationship between the composition of the catalysts and the $|\Delta \log \sigma|$-values, as well as the catalytic reaction yield [2].

The aim of the present research was to investigate the influence of the composition of the catalyst on the $|\Delta \log \sigma|$-values, and also on the catalytic reaction yield for other mixtures of semiconducting oxides. The measurements were carried out with ZnO-Fe_2O_3 mixtures (both oxides are *n*-semiconductors) taken in molar ratio: $4ZnO + Fe_2O_3$, $3ZnO +$

$+Fe_2O_3$, $2ZnO+Fe_2O_3$, $ZnO+Fe_2O_3$, $ZnO+2Fe_2O_3$, $ZnO+3Fe_2O_3$, $ZnO+4Fe_2O_3$ and also with the same ZnO-Cr_2O_3 mixtures (n-semiconductor + p-semiconductor) as used in the previous investigation [3]. The experiments were carried out in the same way as heretofore [1].

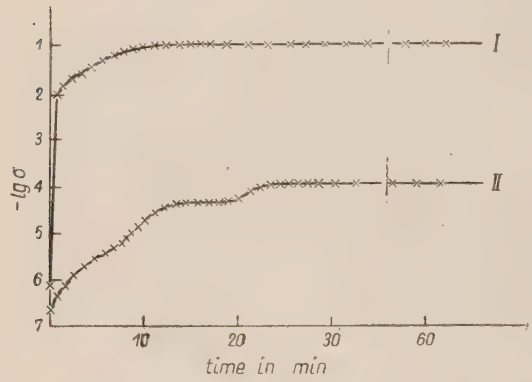

Fig. 1. The change in the logarithm of the electric conductivity of the catalysts in the course of catalytic dehydrogenation of ethyl alcohol.

In the case of ZnO-Fe_2O_3 mixtures, we observed an increase in the conductivity up to a final value which was reached after a few minutes. Curve 1 in Fig. 1 shows the change in the conductivity of the $4ZnO+Fe_2O_3$ mixture measured at 325°C, as a function of time. Such behaviour is characteristic of reaction temperatures above 275°–300°C. At lower temperatures, the change in the conductivity of these mixtures in time is represented by a "step-like" curve, as e. g. curve 2 in Fig. 1, plotted for the $ZnO+2Fe_2O_3$ mixture at 250°.

Fig. 2. shows the $|\Delta \log\sigma|$-values (full line) together with the catalytic reaction yield (dotted line) as the function of the composition of ZnO-Fe_2O_3 mixed catalysts. The diagram corresponds to the measurements carried out at 350°C.

The $|\Delta \log\sigma|$-value slightly increases when we pass from the $4ZnO+Fe_2O_3$ composition to the $3ZnO+Fe_2O_3$ composition; then it decreases down to the $ZnO+3Fe_2O_3$ composition and again slightly increases for the $ZnO+4Fe_2O_3$ composition. The curve representing the catalytic reaction yield shows two local minima (at the $2ZnO+Fe_2O_3$ and the $ZnO+3Fe_2O_3$ compositions) and one local

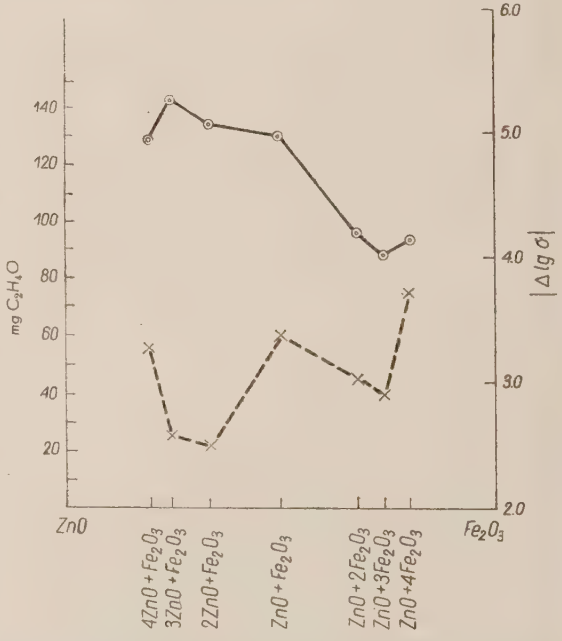

Fig. 2. Dependence of the $|\Delta \log\sigma|$-values (full line) and of the catalytic reaction yield (dotted line) on the composition of ZnO-Fe_2O_3 mixed catalysts. Reaction temperature 350° C.

maximum (at the $ZnO + Fe_2O_3$ composition). This maximum becomes more distinct at higher temperatures. At temperatures lower than 250°, it was not observed at all. At higher temperatures, we observed for the samples of $ZnO - Fe_2O_3$ also a maximum on the $|\Delta \log \sigma|$-composition curve. Nevertheless, the influence of the temperature on the shape of the $|\Delta \log \sigma|$--composition curve is not so marked. This indicates that the changes taking place on the surface of the catalyst, heated in the course of the catalytic reaction, influence its catalytic activity more than its electric conductivity.

With mixtures rich in ZnO, the change in the catalytic activity with the composition has the opposite sign to the change in the $|\Delta \log \sigma|$-value. With mixtures rich in Fe_2O_3, on the other hand, the two values are parallel. It must be emphasised that — when keeping the composition of the sample constant — we obtained linear dependence of $|\Delta \log \sigma|$-values and catalytic reaction yield for all the samples.

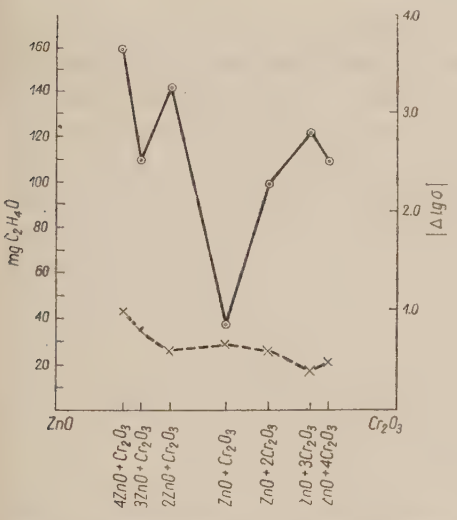

Fig. 3. Dependence of $|\Delta \log \sigma|$-values (full line) and of catalytic reaction yield (dotted line) on the composition of the ZnO-Cr_2O_3 mixed catalysts. Reaction temperature 250°.

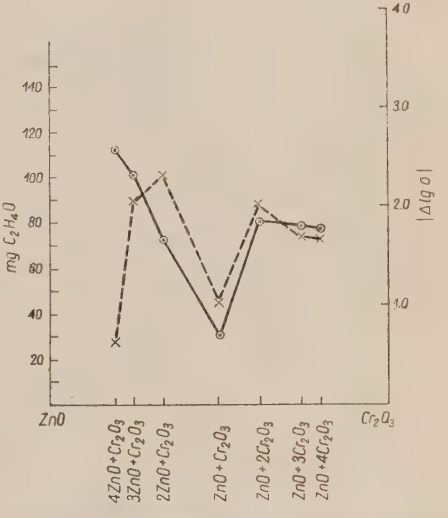

Fig. 4. Dependence of $|\Delta \log \sigma|$-values (full line) and of catalytic reaction yield (dotted line) on the composition of the ZnO-Cr_2O_3 mixed catalysts. Reaction temperature 450° C.

In the case of $ZnO - Cr_2O_3$ mixtures, the curves showing the dependence of the $|\Delta \log \sigma|$-values and of the catalytic reaction yield on the composition of the catalyst change their shape for various temperature ranges. The curves shown in Fig. 3 and corresponding to the measurements carried out at 250°, are characteristic of the temperature range 150°—325°. The full line represents the $|\Delta \log \sigma|$ values and the dotted line, the catalytic reaction yield. The curves shown in Fig. 4 correspond to a temperature of 450°. They give the characteristics of the shape of the

curves obtained for the temperature range 375°—525°. Above 525°, there exists a third temperature region, where the shape of the curves again changes.

In the temperatures intermediate between various temperature regions, we observed a gradual change in the shape of the curves from that characteristic of one region to that characteristic of another.

The results presented in Fig. 3 do not show, for the lowest temperature range, any direct relation between the $|\Delta \log \sigma|$-values and the catalytic reaction yield, taken as the function of the composition of the catalyst. In the 375°—525° temperature range (Fig. 4), the changes in both values show the opposite sign for the samples rich in ZnO ($4ZnO + Cr_2O_3$ to $2ZnO + Cr_2O_3$). For the samples of the $2ZnO + Cr_2O_2$ composition up to $ZnO + 4\,Cr_2O_3$, the changes in the $|\Delta \log \sigma|$-value and in the catalytic reaction yield are parallel. Both curves show a local minimum for the $ZnO + Cr_2O_3$ composition, and a local maximum for the $ZnO + 2\,Cr_2O_3$ composition.

Our present investigation shows that the dependence of the $|\Delta \log \sigma|$-value and of the catalytic reaction yield, both values being taken as the function of the composition of the catalyst, is more complex for the ZnO-Fe_2O_3 and ZnO-Cr_2O_3 systems than in the case of the previously investigated MgO-Cr_2O_3 system.

For each of the systems investigated, we observed different behaviour. This indicates that the relationship between the $|\Delta \log \sigma|$-values and the catalytic reaction yield taken as a function of the composition of the catalyst has a different character for each of the systems.

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CHEMISTRY

Luminescence of Oxide Films Produced During Anodic Oxidation of Aluminium

by

Z. RUZIEWICZ

Presented by M. ŚMIAŁOWSKI on May 21, 1956

Introduction

During anodic direct current oxidation of aluminium in suitably selected electrolyte solutions, a uniform glow of the anodic surface can be easily observed. This phenomenon has repeatedly been the subject of closer studies [1]-[6]. Most authors believed this light emission to be a kind of luminescence connected with the oxide film formed on the anode. There have also been certain tentative attempts to explain it by discharges taking place in the gases evolving at the electrode [3], [7]. It appears that the former hypothesis is strongly supported by the investigations of Gumiński [6] revealing the intensive effect exerted on the luminescence spectrum by admixtures of foreign metals contained in aluminium. However, so far, neither the character of the luminescence in question nor its mechanism have been explained. D. Curie recently mentioned light emission of aluminium anodes [8] with reference to electroluminescence. He does not reckon light emission of anodes among the latter phenomena and states that the electrolytically produced oxide films do not show photoluminescence.

This paper presents the results of certain investigations of the effects of the voltage applied (U) and current density (d) during electrolysis, on the intensity of radiation of aluminium anodes within the visible range of the spectrum (J). The effect of the voltage applied has been examined already [1], [2], [4], but by visual methods and under entirely different conditions. The present paper describes also observations of photoluminescence of anode films.

Experimental part

The anodes were formed in oxalic acid solutions (Merck, pro anal.) of concentrations from 0.8 to 3.0%. This electrolyte was chosen because the glow of anodes formed in it is particularly intensive [6]. Both the anode

and the cathode, each of about 20 sq. cm. in surface area, were cut from pure rolled aluminium foil 65 μ thick. Spectrographic analysis revealed the following impurities: Mn and Fe about 1.10^{-3} per cent; Cu, Mg, Zn, Si, Pb about 10^{-3} per cent each; Ag and Ga about 10^{-4} per cent each *). Before the electrolysis, the electrodes were degreased and washed in redistilled water.

During electrolysis, the electrolyte was cooled with a glass cooler conducting water from a thermostat (15°C) and stirred uniformly. The electrolytic cell was fed from a battery of accumulators through a resistance used as a potentiometer. Regulation made it possible to maintain a constant intensity of the current (± 1 per cent) or a constant voltage (± 0.4 per cent), while determining the dependencies $J = f(U)$ or $J = \varphi(d)$ respectively.

It is known [9] that when anodic films are formed in oxalic acid at a constant current density greater than about 10 mA/cm²., the initial voltage rises swiftly, then drops and begins finally to rise slowly; this moment marked the beginning of $J(U)$ measurements. To conduct measurements at a constant voltage the anodes were formed up to the required voltage, which was then maintained. Initially, the current density dropped rapidly at a constant voltage and after some time progressively slower. The readings were stopped when oxidation took place through the entire thickness of the foil.

A highly insulated vacuum photoelectric cell and Wulf's electrometer (of a sensitivity adjusted within 60—40 mV/dash) were used for measuring by the stationary deflection method [10] the intensity of the light radiated by the anode. The resistance for measuring the potential drop was about $2.8 \cdot 10^{12}$ ohms; the exact value of the resistance was measured in each series of readings and the photoelectric currents were calculated. They were within the limits of 10^{-13}—10^{-12} A. Also, the "dark current" of the photoelectric cell was measured each time.

A high-pressure mercury discharge lamp, with ultraviolet filters or with a quartz monochromator, was used for exciting photoluminescence of dried anodic films.

Results and discussion

Fig. 1 represents the dependence $J = f(U)$ at $d = \text{const.}$ for several cases. It can be seen that J rises linearly with U, within a certain range. At higher voltages, J rises less rapidly, and at still higher voltages, J ceases to rise and even drops occasionally. The higher the concentration of the acid, the sooner deviation occurs from the rectilinear course, a fact which appears to indicate a correlation with the oxide layer dissolution favoured by high acid concentrations as well as by a rise of temperature induced within the layer by higher voltages [7], [9]. The rise of temperature may also

*) The author is greatly indebted to A. Idzikowski, for his assistance in determining the amount of impurities.

be responsible for the decrease for luminescence. Similar correlation between the intensity of the light emitted and voltage was described for the electroluminescence of ZnO-rectifying elements [11].

Several curves illustrating the dependence $J = \varphi(d)$ at $U = \text{const.}$ are represented in Fig. 2. As d diminishes, J drops, first slowly and then more rapidly and almost linearly. In order to explain this fact, it may be remarked that the total current passing through the oxide layer consists of ionic- and electron-currents [12], [13]. At a constant voltage, the ionic current diminishes more rapidly [14] and is principally responsible for the initial drop in current density. Since, on the other hand, the anode glow can be connected only with the transfer of electrons [4]-[6], the diminishing density of the current rapidly diminishes intensity of the light only when the electron current constitutes a relatively large part of the total current passing through the layer. Quantitative evaluation would be somewhat difficult since the participation of the electronic current in the total passing through the layer can be calculated only when formation takes place in electrolytes not dissolving oxide films [14]. The shape of the curves obtained may also be affected in some degree by absorption of the light in the growing film.

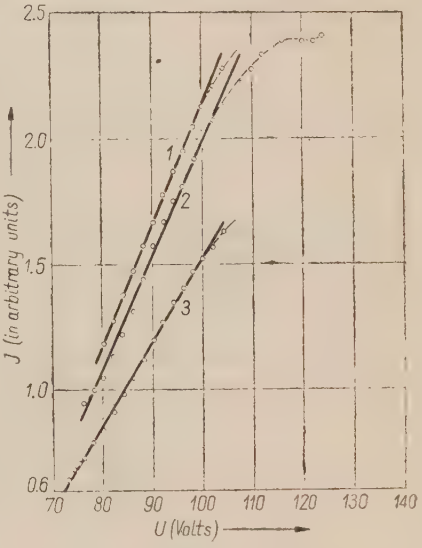

Fig. 1. $J = f(U)$. $1 - d = 27.2$ mA/cm². $2 - d = 25.0$ mA/cm². $3 - d = 20.8$ mA/cm². 1 per cent $(COOH)_2$

The oxide films produced under the conditions described show, after washing and desiccation, a bluish fluorescence growing considerably in intensity after heating for 2 to 3 hours at a temperature of about 400° C. in air or *in vacuo*. Fluorescence was excited by long-wave ultraviolet irradiation (from 302 to 366 mμ). The spectrum is of the band type with the greatest intensity in the blue-green region, similarly to the maximum in the anode glow during electrolysis [6]. After stopping the exciting ultraviolet irradiation, afterglow can also be observed for a few seconds; unlike fluorescence, it has a yellowish colour. These facts are in disagreement with D. Curie's statement quoted in the introduction to the present paper.

It appears that the aluminium anode glow observed can be regarded as an electroluminescence phenomenon, since it has a number of qualities resembling the electroluminescence of other systems endowed with rectifying properties. It should be noted that ZnO layers as well as SiC crystals

show intensive luminescence when a solution of the electrolyte constitutes the cathode in the rectifying elements produced of either of the two substances mentioned [11], [15]. They are also known to show photoluminescence. The mechanism of electroluminescence of aluminium anodes

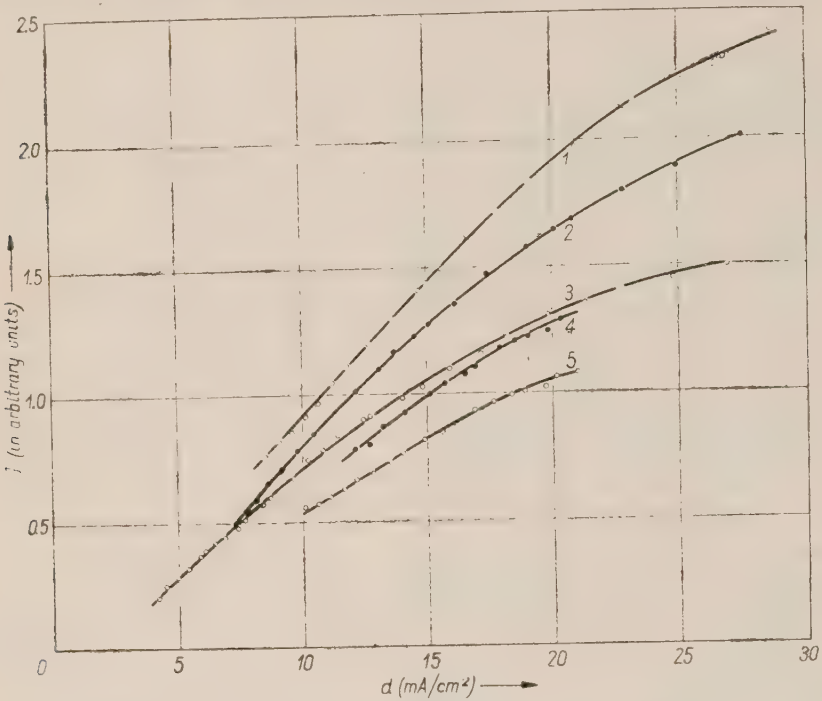

Fig. 2. $J = \varphi(d)$ · 1 — U = 110 V 4 — U = 90 V
2 — U = 98 V 5 — U = 80 V
3 — U = 90 V 1.5 per cent $(COOH)_2$
0.8 per cent $(COOH)_2$

needs further investigation. The mechanism is no doubt complicated by the simultaneous formation and excitation of the same active layer, taking place during anodic polarisation. Not the entire anodic film but probably only its thin non-porous layer in direct contact with the anode ought to be regarded as responsible for luminescence; the electric field strength is, during electrolysis, particularly high in this layer, attaining 10^7 V/cm. [4], [5]. According to Güntherschultze [12], ionisation by impact takes place in this layer during the passage of electrons, and in many cases this ionisation is considered as a reason of electroluminescence [11], [16].

The author is greatly indebted to Professor K. Gumiński for the profound and continued interest he took in this work, as well as to K. Pigoń, for much valuable advices and discussions.

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CHEMISTRY

Luminescence of Aluminium Electrodes During Electrolysis by Alternating Current in Solutions of Oxalic Acid

by

Z. RUZIEWICZ

Presented by M. ŚMIAŁOWSKI on May 21, 1956

Introduction

Not only direct current but also alternating current may be used for the formation of oxide films on aluminium electrodes. When both electrodes of the electrolytic cell are made of aluminium, each becomes covered with an oxide film developing in the half-periods of this current during which the given electrode is anodically polarised [1]. Both electrodes emit radiation during electrolysis, but the spectral distribution of light differs from that emitted by aluminium anodes formed in the case of direct current. Qualitative determinations by T. Rummel demonstrated that the maximum of the intensity shifts during a. c. electrolysis towards the red end of the spectrum. Stroboscopic observations by the same author revealed also that the electrode emits light in the positive as well as in the negative half-periods, but the colour is different in the two half-periods [2]. These results were taken by Rummel to be confirmation of the thesis that aluminium electrode glow (also in the case of d. c. electrolysis) should be attributed to discharges in the gases [3]. Since, however, there is a number of arguments suggesting that aluminium anode glow is a luminescence excited in the oxide film under participation of electrons passing through it during electrolysis [4], it appeared worth while to investigate aluminium electrode glow more closely during a. c. electrolysis, as well as to make an attempt at explaining the phenomena observed in agreement with views presented earlier [4].

Experimental part

The material used for electrodes was in this work the same as in the investigations described earlier on the luminescence of aluminium anodes during d. c. electrolysis [4]. The same applies to the electrolyte, conditions of formation, and to methods of determining the dependence of the

visible-light intensity (J) on the voltage applied (U) and current density (d). The only essential difference was that the electrolytic cell was fed with a. c. (50 c./sec.) or from a rotating commutator generating rectangular impulses of controlled frequency. A small Krüss spectrograph with glass optics (dispersion 75 A.U./mm. at $\lambda = 5{,}000$ A.U.) and F. P. Ultrapan-Super plates were used for preliminary spectrographic examinations. An MFII microphotometer was used for measuring the density of the plates.

Results and discussion

A. c.-formed pure aluminium electrodes emit a yellowish light while analogous anodes formed with d. c. emit, under otherwise identical conditions, a bluish light. In the first case a broad band in the yellow-red region, with the peak at about 595 $m\mu$., predominates in the luminescence spectrum (Fig. 1). In the second, according to Gumiński [5], the peak of anodic emission of pure aluminium lies below 500 mμ. It may be remarked, with reference to Rummel's views quoted in the introduction to the present paper, that the spectra obtained resemble the spectra of crystal phosphors luminescence rather than those of discharges in gases.

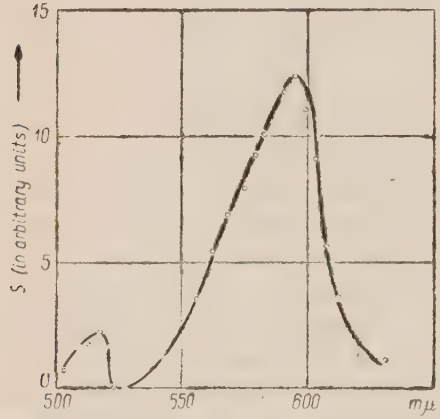

Fig. 1. Microphotometric curve of the spectrum of an aluminium electrode formed by alternating current. S – density. Formation in 1.5 per cent $(COOH)_2$ at $U = 100$ V. Duration of exposure 3 hours.

The dependencies $J = f(U)$ at $d = \text{const.}$ (Fig. 2) *) and $J = \varphi(d)$ at $U = \text{const.}$ (Fig. 3) for the glow appearing in a. c. electrolysis also take an entirely different shape from those in the case of d. c. electrolysis (cf. [4]). Without entering into a more detailed analysis, it ought to be stressed that, as in the case of the emission spectra, the dependencies bear evidence of essential differences in the mechanism of the light emission of aluminium electrodes in d. c. and a. c. electrolysis.

More detailed information was obtained by stroboscopic observation of the electrodes at an a. c. frequency from 2 to 20 c./sec. It was established that the electrode emits, during the positive (anodic) half-period, a uniformly bluish light closely resembling that emitted by anodes during d. c. electrolysis. On the other hand, in the negative (cathodic) half-

*) The graphs refer to a range of voltages other than that in graphs of analogous dependencies for anodic luminescence published earlier [4]. This is because in a. c. electrolysis the voltage starts to rise continuously beginning with much higher initial values than in d. c. electrolysis (at identical current densities).

period, the luminous emission takes on the character of yellow-red flashes; the intensity of the flashes is greatest immediately after the change from the positive to the negative half-period, and rapidly decreases so that at low frequencies the electrode does not glow at all at the end of the cathodic half-period. Single flashes were found to last about 0.05 sec. (at a frequency of 2.5 c./sec. and $U = 105$ V). Consequently, when, observed directly and at low frequencies (a few c./sec.), the electrode is seen to emit a bluish light interrupted by brief yellow-red flashes. At higher frequencies (50 c./sec.), on the other hand, the electrode appears to glow without interruption in a yellowish colour. These experiments demonstrate that the differences in the luminescence of aluminium anodes, as between d. c. and a. c. electrolysis, have their origin in the fact that in the latter

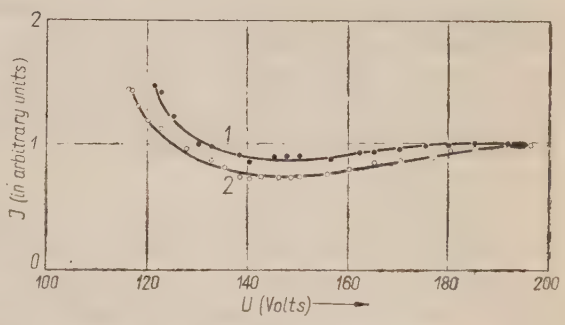

Fig. 2. $J \quad f(U)$. $1 - d = 25.5$ mA/sq. cm.; $2 - d = 20.1$ mA/sq. cm.; 1 per cent $(COOH)_2$

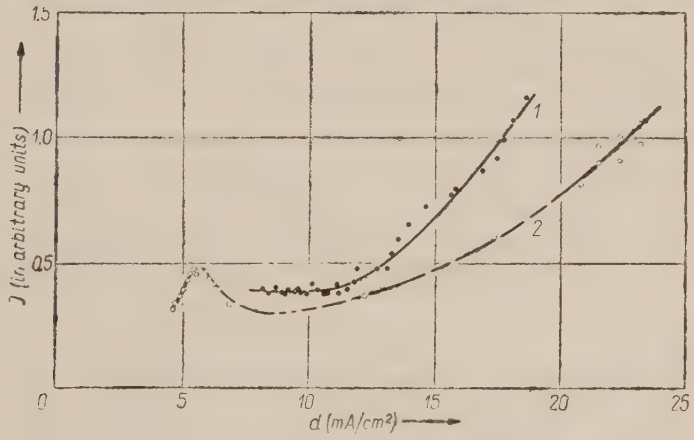

Fig. 3. $J = \varphi(d)$. $1 - U = 105$ V, 1.5 per cent $(COOH)_2$; $2 - U = 125$ V, 1 per cent $(COOH)_2$.

case the flashes induced by the change of polarisation from anodic to cathodic are superposed on anodic luminescence.

The cathodic flashes described have been known before ([3], [6]), but have not been studied more closely, nor was attention given to their relation to the qualities of the glow of aluminium electrodes formed with a. c.

In the course of this work, the origin of the flashes was explained by a number of experiments which demonstrated the following:

1° — The cathode flash occurs only when preceded by anodic polarisation of the electrode. 2° — Cathodic polarisation fails to cause a flash when the electrode has been irradiated with long-wave light before the voltage-direction change. 3° — A brief flash, analogous to the cathodic flash, can be produced by rapidly heating the electrode to about 300° C. immediately after termination of anodic polarisation. 4° — A similar flash can be produced by heating an electrode covered with an oxide film after ultraviolet irradiation and after the end of the afterglow occurring at room temperature.

All the above phenomena can be explained by treating the oxide film formed on the electrode as crystal phosphor, and applying to it the phosphors band model. In such a model, we must assume the presence of local electron levels (traps) situated at various depths below the conduction band. These levels are filled with electrons transferred to the conduction band either by ultraviolet irradiation or during anodic polarisation. Luminous emission can take place owing to radiating transitions during which electrons, liberated from trap levels, lose their energy. Thermal energy corresponding to room temperature suffices to liberate electrons from the shallow levels; this is the origin of phosphorescence *).

Release from deep traps is due to higher temperature or to long-wave radiation or - as shown in the experiments referred to above — cathodic polarisation. In the latter instance, liberation of electrons occurs, perhaps, by non-elastic collisions with electrons travelling from the metal to the film. It has long been known (B. Gudden and R. W. Pohl [7]) that electrons can be released from the traps of phosphorescent substances with the aid of an electric field, the release being accompanied by a slowly decaying flash. The cathodic flash might thus represent a variation of this effect, the peculiarity of which is that the traps are first filled with electrons not by irradiation but by anodic polarisation of the oxide film.

It should be noted that the presence of local levels filled with electrons during anodic polarisation has been taken into consideration in connection with the negative space charge generated in oxide films [8]. A concept similar to the mechanism proposed for the cathodic flash was advanced with a view to explaining the hysteresis loop in the current-voltage characteristic of the oxide films [9]. Thus the mechanism proposed for the cathodic flash explains the luminous phenomena described, and is also in agreement with investigations of the electric properties of the layers. Further investigations, with a view to explaining the mechanism of anodic luminescence and of the transitions connected with light emission are in progress.

*) Phosphorescence fails to appear after excitation of the film by anodic polarisation at room temperatures, which is easily explained by instantaneous release of electrons from the shallower traps, taking place in the electric field [10].

The results presented in this paper appear to support the original assumption [4] that luminescence of aluminium anodes during electrolysis ought to be regarded as electroluminescence.

The author is greatly indebted to Professor K. Gumiński, for the profound interest taken in this research, and to K. Pigoń as well as to J. Rohleder for their valuable suggestions and discussions.

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The Ludlow Deposits in Eastern Poland

by

H. TOMCZYK and L. TELLER

Communicated by J. SAMSONOWICZ at the meeting of May 18, 1956

Deep boring carried out in 1955 at Chełm on behalf of the Geological urvey of Poland revealed, at a depth of 1207.5 m., Silurian deposits mmediately underlying those of Visean age. At a depth of 1607.4 m., enetration of these deposits was not achieved.

Lithologically, the Silurian strata present a monotonous series of rgillaceous shales with several thin limestone intercalations. The shales re calcareous, of light-grey colour, bearing considerable amounts of nica and traces of pyrite; they are silky to the touch, indicating weak etamorphism. In the shale concretions of limestone are abundant, up 10 cm. in diameter, of elliptic shape and bearing no macrofauna.

The bedded limestone intecalations are recorded within depths om 1255.9 to 1259.0 m. and again from 1363.4 to 1384.9 m. Within he upper occurrence are three intercalations, two of them having a thic- ness of 10 cm. each; the third intervenes and has a thickness of 70 cm. he lower occurrence contains two thicker intercalations of 70 and 80 cm. espectively, and two thin ones of 10 cm. each. Both bedded and con- retionary limestones are finely crystalline, dark-grey, with an admixture f argillaceous matter, somewhat bituminous, with traces of pyrite.

Graptolites predominate in the fauna, occurrence in the profile not eing uniform. The particular rhabdosomes are not always in a satisfactory tate of preservation. Few forms are encountered in the top part of the rofile within depths from 1209.4 to 1213.8 m., while between 1213.8 nd 1225.6 m. they are altogether wanting. They reappear, however, nough in great scarcity, between 1225.6 and 1234.4 m. From that depth own to 1362.7 m., the graptolites are again completely lacking, the Tautiloids being here the only faunal representatives. It is only within epths from 1362.7 m. down to the bottom of the profile at 1607.4 m. nat graptolites are so copious as sometimes to coat the entire core surface.

Pelecypods also are abundant in the shales. Such genera as *Cardiola*, *vicula*, *Lunulicardium*, *Dualina*, *Aviculopecten*, *Vlasta*, *Astarte*, *Anti- leura* and others have been differentiated among them. Orthoceracones

are also somewhat common, but strong compression makes them unindentifiable, even generically. Representatives of other fauna, mostly in the upper portion of the profile, consist of small numbers of trilobites, ostracods and brachiopods, the latter occurring only very sporadically; there are also impressions of fish fragments and numerous problematica which call for more detailed investigation.

Nautiloids with re-crystallised shells are extremely abundant in bedded limestones; *Cardiola*, *Avicula*, *Dualina* and other forms of pelecypods somewhat less so, while ostracods and ichthiodorulites are quite scarce.

The entire Silurian series displays an undisturbed, almost horizontal bedding, the only deformations being a dozen or so slide planes, the inclination angle of which ranges from 10 to 45°. Close to these planes, the shales usually show strong compression and mylonitisation. There also occur numerous calcite veinlets, a mere few millimetres in thickness and fingering out in various directions.

Pristiograptus ultimus Perner and other species of this genus are seen crowding in the basal portions of the core. In the Silurian from Bohemia, this species is characteristic of the lowest zone of the Přidole limestones in the Budňany beds [6], [11], [12]. In the central part of the graptolitic series, the Silurian index forms of Bohemia recorded as occurring in large numbers are *Monograptus bouček*i Přibyl and *Monograptus perneri* Bouček. In Bohemia [6], [11], [12] these forms are common enough in the middle zones of the Přidole beds. The top parts of the core have yielded a small number of the index forms *Pristiograptus transgredieus* Perner and of those probably referable to *Monograptus angustidens* Přibyl, both of which in the Ludlow of Bohemia are characteristic of the upper part of the Přidole beds. At present, it is hardly possible to establish whether the Upper Ludlow zones are representend at Chełm, i. e., whether the zones of *Monograptus uniformis* Přibyl and *Monograptus hercynicus* Perner occur.

Below the *Pristiograptus ultimus* zone, an abundant graptolitic fauna is also recorded from Chełm, but *Saetograptus leintwardinensis* has not been discovered there. Possibly these are graptolites characteristic of the uppermost Lower Ludlow beds, i. e., the *Pristiograptus fragmentalis* [6] zone.

It should be stressed that *Linograptus* sp., occurring throughout the whole of the Silurian series at Chełm, belongs to the common but characteristic forms of the Middle Ludlow also recorded from the Ludlow of Bohemia [13], the Ockerkalk of Thuringia [7], [11] and the vicinity of Chernovtsy [17].

By their graptolitic fauna, the Ludlow deposits from Chełm are thus seen to agree very closely with those from Bohemia. They differ considerably, however, as regards lithologic developement, since shales,

only quity insignificantly intercalated by limestones, are here predominant, whereas tabular limestones with shale intercalations predominate in Bohemia. Only in the two uppermost Middle Ludlow zones of Bohemia, i. e., in those of *Pristiograptus transgrediens* and *Monograptus angustidens*, do shales play a more important part [6].

Intensity of sedimentation is another feature differentiating the Ludlow beds of Chełm from those of Bohemia. Whereas in Bohemia (and in Thuringia) the thickness of the Ludlow beds never exceeds a few

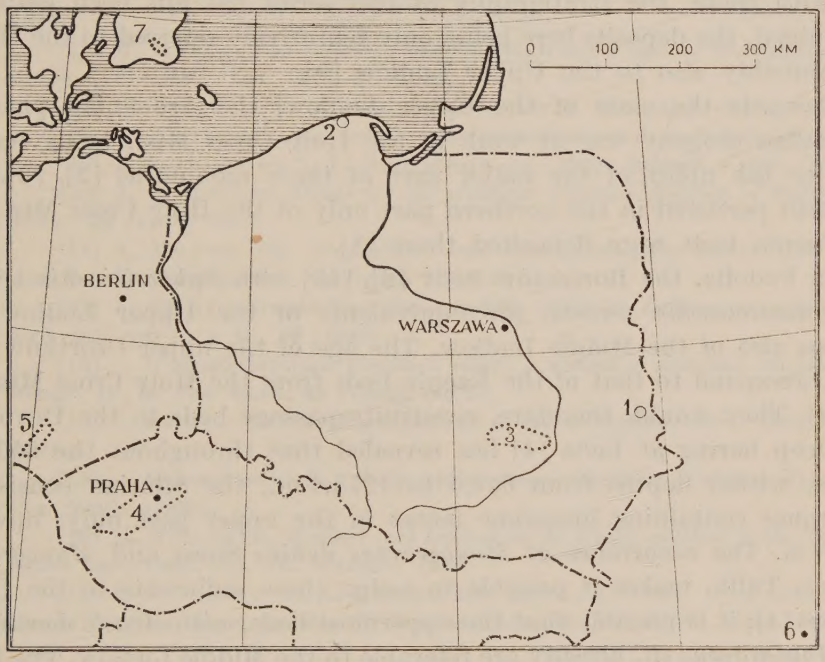

Fig. 1. Sketch map of localities with Ludlow deposits: 1 — Chełm, 2 — Łeba, 3 — Holy Cross Mts., 4 — Barrandian, 5 — Thuringia, 6 — Chernovtsy, 7 — Ramsåsa

dozen metres, at Chełm the Middle Ludlow deposits — with the upper part of the Lower Ludlow, also possibly included — were not penetrated by boring drilled through the 400 m. series.

The discovery of a graptolite facies in the Middle Ludlow in the east of Poland is of considerable significance for the palaeogeography of that stage and for faunal migration problems. Middle and Upper Ludlow graptolites have been recorded from the Silurian of Bohemia only [2], [13]. In Thuringia [7], only the Lower and Upper Ludlow horizons have been differentiated on evidence supplied by graptolites, white Middle Ludlow beds represented by limestones (Ockerkalk) do not yield graptolites except representatives of genus *Linograptus*, which are diagnostic in determining their stratigraphy.

Considerable facial differences are displayed in the development

of the Middle Ludlow deposits from Chełm, as compared with those from the Holy Cross Mts. [2], [14]. In the eastern area of the Holy Cross Mts. the graptolitic series reaches the *Pristiograptus nilssoni* [16] zone, and in the central area the *Saetograptus leintwardinensis* zone. Above these zones, the sediments gradually grow terrigeneous and are represented by a series of greywacke shales. Besides other forms, they yield *Camarotoechia nucula* and *Sphaeririnchia wilssoni* as well as some forms of genus *Pristiograptus*, extremely rare and closely unidentifiable. In connection with these, the stratigraphy of this series has not been definitely determined, the deposits here being only tentatively referred to the Middle and probably also to the Upper Ludlow [9].

Towards the close of the Upper Ludlow, the Ardennian phase of Caledonian orogeny was at work in the Holy Cross Mts., being responsible for the uplift of the major part of these mountains [2], [9]. The sea basin persisted in the northern part only of the Holy Cross Mts., and the Rzepin beds were deposited there [3].

In Podolia, the Borszczów beds [8], [15] with *Sphaeririnchia wilsson* and *Camaroteochia nucula*, are equivalents of the Upper Ludlow and perhaps also of the Middle Ludlow. The age of the upper Czortków beds may correspond to that of the Rzepin beds from the Holy Cross Mts. [2], [8], [9]. They would, therefore, constitute passage beds to the Devonian.

Deep boring at Łeba [4] has revealed that throughout the 600 odd metres, within depths from 669.5 to 1273.4 m., the Silurian consists of mudstones containing limestone lenses in the upper part only, upwards of 705 m. The occurrence of *Monograptus dubius* Suess and *Monograptus scanicus* Tullb. makes it possible to assign these sediments to the Lower Ludlow [4]; it is possible that the uppermost beds, with *Acaste downingiae* and *Monograptus* sp., already are referable to the Middle Ludlow. The facies of Silurian beds from Łeba [4] have been said to bear similarities with that of the Ramsåsa formation of Scania. The latter formation, however, consisting of sandstones and shales, frequently red-coloured, does not bear graptolites [5], [10]. The sedimentation of the Ludlow beds at Łeba was accompanied by rapid and extensive subsidence of the sea bottom, resembling environmental conditions of Chełm. The latter area was, however, under more pelagic conditions.

Data thus far available seem to prove the existence in the Middle Ludlow of a marine basin bounded by the Holy Cross Mts. and Podolia, which were then situated in the littoral belt, and by Łeba. The deepest parts having bathyal conditions at that time seem to have been in the vicinity of Chełm. The basin had a broad connection with the Barrandian area, where existed a sea, more shallow but open.

Additional research studies will be needed to determine the directions of faunal migrations across the various parts of the Ludlow sea and to acquire a better knowledge of its palaeogeographic problems.

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